Ph.D. Qualifying Exam B

Statistical Inference

May 25, 2022

1. **(20 points)** Suppose that $Y_1, \dots, Y_n$ is a random sample from a binomial probability distribution with parameters k and p. Let $\tau(p) = kp(1-p)^{k-1}$. Find the MVUE of $\tau(p)$.

2. Suppose that a random sample of four observations is obtained from the density function

$$f(y;\theta) = \left(\frac{1}{2\theta^3}\right) y^2 e^{-y/\theta} I(y > 0).$$

   (a) **(10 points)** Find the (generic) rejection region of the most powerful test of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

   (b) **(6 points)** Find the explicit rejection region of the most powerful test you derived in part (2a), when $\theta_0 = 2$ and $\alpha = 0.05$. Note: Finding the distribution of $2Y_i/\theta$ may help.

   (c) **(4 points)** Is the test uniformly most powerful for the alternative $\theta > \theta_0$? Give clear justification for your answer.

3. Let $Y_1, Y_2, \dots, Y_n$ be a random sample from a population with density function

$$f(y|\theta) = \begin{cases} \frac{3y^2}{\theta^3}, & \text{if } 0 < y \le \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) **(5 points)** Find the maximum likelihood estimator of θ with adequate justification.

   (b) **(5 points)** Find a sufficient statistic for θ. Show details.

   (c) **(5 points)** A pivotal quantity is defined as a function of the data and the parameter θ but whose distribution is free of the parameter. For example, if $X \sim N(\mu, \sigma^2)$, standardizing X gives a pivotal quantity. Obtain a pivotal quantity for θ.

(d) **(5 points)** Use the pivotal quantity from part (3c) to derive a $100(1-\alpha)\%$ (two-sided) confidence interval for θ.