Doctoral qualifying exam questions on Math 651 for May 2022

1. Given the nonlinear system of ODEs

$$\begin{aligned} \dot{x} &= 2x + x^2 + y^2 \\ \dot{y} &= y - xy \end{aligned}$$

(a) Find the equilibrium or critical points, then linearize about each critical point to determine its type (e.g., saddle, spiral, etc.). Sketch the phase portrait of each linearized system and write down its general solution.
(b) Find the null clines, on which $\dot{x} = 0$ or $\dot{y} = 0$, and sketch them in the x, y-plane.
(c) Sketch the flow or phase portrait of the nonlinear system. Describe how the solutions behave for large time.

2. **(a)** For the differential equation

$$x^2 y'' - 4xy' + 6y = x^4 \sin x$$

show that $y_1 = x^2$ is a solution of the associated homogeneous equation, then use reduction of order to find the general solution of the *in*homogeneous equation.
(b) For the differential equation, known as Legendre's equation,

$$(1 - x^2)y'' - 2xy' + a(a+1)y = 0$$

where $a > -1$ is a real parameter, find and classify (as regular or irregular) the singular points of the ODE in the finite part of the complex plane.

The point $x = 0$ is an ordinary point of the ODE (why?). Set $y = \sum_{n=0}^{\infty} a_n x^n$ and find the series solution for each of two linearly independent solutions about $x = 0$. Briefly explain why, if $a = 2m$ is an even integer or if $a = 2m+1$ is an odd integer (where $m = 0, 1, \ldots$), the series for one solution terminates to give a polynomial. What happens to the other series solution, and for what x do you expect it to converge?

3. Consider the problem for Laplace's equation on a half-annulus $0 < a < r < b$, $\theta \in (0, \pi)$ given by

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$
$$u(r,0) = u(r,\pi) = 0, \quad u(a,\theta) = f(\theta), \quad u(b,\theta) = g(\theta)\,.$$

Look for a separable solution of the form $u(r,\theta) = X(r)Y(\theta)$. Note that the boundary conditions on $\theta = 0$ and $\theta = \pi$ are homogeneous, find the corresponding eigenvalue problem and state its solution. Hence find the solution for $u(r,\theta)$ as a series for general $f(\theta)$ and $g(\theta)$.

Give the solution explicitly when $a = 1$, $b = 2$, $f(\theta) = \sin\theta$ and $g(\theta) = 2\sin 3\theta$. Briefly explain how you would solve the problem if the boundary conditions on $\theta = 0$ and $\theta = \pi$ are *in*homogeneous, that is, $u(r,0) = \phi(\theta)$ and $u(r,\pi) = \psi(\theta)$.