Doctoral qualifying exam questions on Math 651 for January 2023

1. (a) Find the general solution of the 2D linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Sketch the direction field and describe the type (e.g., saddle, etc.) of the equilibrium point at the origin.

(b) Find the general solution of the homogeneous differential equation

$$L(y) \equiv 2x^2 y'' + xy' - 3y = 0\,.$$

Then: (i) find the solution of the initial value problem with $y(1) = 1$ and $y'(1) = 4$. (ii) Find the inhomogeneous solution of the corresponding inhomogeneous equation $L(y) = 1+x$.

(c) Show that $y = x$ is a solution of the differential equation

$$x^2 y'' - x(x+3)y' + (x+3)y = 0\,,$$

then use reduction of order to find a second linearly independent solution.

2. (a) Find and classify, as regular or irregular, all singular points in the finite part of the complex plane for the ODE

$$x(x-1)^2 y'' + (x-2)y' + (x-2)y = 0\,.$$

What is the leading (or dominant, one-term) behavior of the two linearly independent solutions of the ODE as $x \to 0$?

(b) Explain why the point $x = 0$ is an ordinary point of the ODE

$$(1+x^2)y'' - 4xy' + 6y = 0\,.$$

Look for power series solutions about $x = 0$ of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Find the recurrence relation for the coefficients $\{a_n\}$, and find the first three non-zero terms of two linearly independent solutions (unless the series terminate sooner).

3. Consider the problem for the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin\left(\frac{\pi x}{L}\right) e^{-t} \qquad x \in (0, L) \quad t > 0\,,$$
$$u(0,t) = c_1\,, \quad u(L,t) = c_2\,, \quad u(x,0) = f(x)\,.$$

where c_1 and c_2 are nonzero constants.

(a) Put $u(x,t) = u_e(x) + v(x,t)$ where $u_e(x)$ is a steady, large-time ($t \to \infty$) or equilibrium solution and $v(x,t)$ is time-dependent or transient. Write down the problems that determine $u_e(x)$ and $v(x,t)$, and find the solution of the problem for the steady component $u_e(x)$.

(b) Find the eigenvalue problem for a separable solution $v(x,t) = X(x)T(t)$, and write down the eigenvalues and eigenfunctions $(\lambda_n, X_n(x))$. Now look for a solution of the form $v(x,t) = \sum_n^\infty a_n(t)X_n(t)$, and find the problem that determines $\{a_n(t)\}$. Find the solution of this problem for $\{a_n(t)\}$. Find the leading nonzero or dominant behavior of $v(x,t)$ as it tends to zero when $t \to \infty$. Explain how this can change with L.