Applied math written qualifying exam, Jan 2023

January 8, 2023

1. Consider the following random walk. At $t = (n-1)\Delta t$ a particle is located at $x = m\Delta x$. At $t = n\Delta t$ the particle moves to $x = (m+1)\Delta x$ with probability p_r, to $x = (m-1)\Delta x$ with probability p_l, and it will vanish (be irreversibly lost from the system) with probability p_s. Let $w(m, n)$ denote the probability the particle is at $x = m\Delta x$ at time $t = n\Delta t$.

(a) Draw a grid indicating the achievable positions a particle can reach at time $t = n\Delta t$, for $n = 0, \ldots, 3$. Determine the probability for all the achievable positions in the grid.
(b) What are the values of A and B such that $w(m, n) = Aw(m-1, n-1) + Bw(m+1, n-1)$?
(c) Set $p_r = p_l = p$ and $u(x, t) = w(m, n)$ and derive a PDE for u, assuming Δx, Δt are small. In doing this, assume the probability of loss is small, i.e., $p_s = p_0\Delta t$. Also, set $D = \Delta x^2/\Delta t$.
(d) Write an expression for the probability the particle disappears by $T = n\Delta t$ (as $\Delta t \to 0$ with T fixed) in terms of p_0 and T. Hint: The probability the particle is still on the grid at time n is $(1-p_s)^n$.

2. This question is on disease modeling. Let S be the subset of a population susceptible to a disease, I is the subset that is ill and infectious, and R is the subset that is recovered. Assume that anyone who is ill will eventually recover, and that the immunity of a recovered person eventually wears off, at which point they again become susceptible.

(a) Write a set of 3 reaction equations and a set of 3 kinetic equations (ODEs) for S, I, and R. Let k_i be the reaction rate of equation i.
(b) Find a single conservation law, and use it to reduce the system to two equations for S, I and the conserved quantity.
(c) Find two steady states, and interpret them physically (i.e., in terms of the disease progression). Determine the stability of the steady states.

3. This is a traffic modeling problem in which cars are allowed to enter and exit the highway.

(a) Assume that over an interval $x_0 - \Delta x < x_0 + \Delta x$ the number of cars that enter or exit from $t = t_0 - \Delta t$ to $t = t_0 + \Delta t$ is $4\Delta x\Delta t Q$, where $Q(x, t)$ is the net rate per unit length at which cars are entering or leaving the highway. Show clearly that the balance law for traffic flow is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x} + Q,$$

where $\rho(x, t)$ is the density and $J(x, t)$ is the flux (cars passing x per unit time).

(b) Assume the constitutive law $Q = \alpha(\rho - \beta)$, where α and β are constants. Is there any (physical) reason to assume α is either positive or negative?
(c) Assume a constant velocity $v = v_0$ and solve the equation with the constitutive law in (b) and initial data $\rho(x, 0) = e^{-x^2}$ by integrating along characteristics (or other means). Note the density is no longer constant along characteristics. What is the effect of Q on the density?