PhD Qualifying Exam Problems: Math 613 (Math Modeling)

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Problem 1

Consider the planar system given in polar coodinates by

$$\dot{r} = \alpha r - r^3,$$
$$\dot{\theta} = \sin\theta + \beta.$$

(a) Find the conditions on parameters α and β under which the system has (1) one equilibrium point and no periodic orbits, (2) two equilibrium points and no periodic orbits, and (3) no equilibrium points and one periodic orbit. Sketch the phase portrait for each of the cases.
(b) Show that for a fixed $|\beta| > 1$ the system undergoes a Hopf bifurcation at $\alpha = 0$.
(c) Describe bifurcations that occur for fixed $\alpha > 0$ when β is varied. Sketch the corresponding bifurcation diagram.

Problem 2

Consider the diffusion equation

$$u_t = u_{xx},$$

with zero Dirichlet boundary conditions imposed on $x = 0$ and $x = 1$ as well as initial condition $u(x, t = 0) = x$. Solve the PDE by using the method of separation of variables, applying the boundary and initial condition.

Problem 3

(a) Show that the eigenvalue problem

$$\begin{cases} -f''(x) = \lambda f(x), \\ f'(0) = a_0 f(0), \quad f'(L) = -a_L f(L), \end{cases}$$

cannot have negative eigenvalues when $a_0 > 0$ and $a_L > 0$.
(b) Show that the eigenvalue problem

$$\begin{cases} f''''(x) = \lambda f(x), \\ f(0) = f(L) = f''(0) = f''(L) = 0 \end{cases}$$

cannot have negative eigenvalues.