Math

Name: ______________________________

Instructions: Show all work and justify all steps of each argument you make. Points may be deducted if either is missing or inadequate. Note that there is one question on the back. You have $2\frac{1}{2}$ hours for this exam.

1. (a) Show that the function $g(x) = x + c\,(1 - \tan(\pi x))$ has a fixed point $x^* = \frac{1}{4}$ and determine for which values of c the fixed point iteration converges when the starting guess x_0 is chosen sufficiently close to x^*.

   (b) Discretize the boundary-value problem

$$\begin{aligned} y''(x) &= e^{-y}, 0 < x < 1 \\ y(0) &= 2,\ y(1) = 1, \end{aligned}$$

   with a stepsize $h = \frac{1}{3}$ and set up but do not solve a Newton iteration, explicitly calculating all the terms needed to set up the iteration.

2. Heun's method for the solution of the initial value problem $\frac{dy}{dt} = f(t, y)$ takes the form

$$\begin{aligned} k_1 &= f(t_i, y_i); \\ k_2 &= f\left(t_i + \frac{2h}{3}, y_i + \frac{2h}{3}k_1\right); \\ y_{i+1} &= y_i + \frac{h}{4}(k_1 + 3k_2). \end{aligned}$$

   (a) Show that the local truncation error of this method is second order.

   (b) Find an inequality satisfied by $z = h\lambda$ that defines the region of absolute stability for this algorithm and find the interval on which this is satisfied for $\lambda \in \mathbb{R}$.

3. (a) So-called *open Newton-Cotes formulas* are quadrature rules based on polynomial interpolants of degree n defined on a uniformly spaced grid on the interval $[0, 1]$ such that the values at the endpoints are not used to define the interpolant.

      i. Show that for $n = 2$, the formula is given by *Milne's method*

$$\int_0^1 f(x)dx = \frac{1}{3}\left(2f(x_0) - f(x_1) + 2f(x_2)\right) + e_{\text{Milne}}.$$

      for $f \in C^3([0,1])$ with $h = \frac{1}{4}$ (more generally $h = \frac{1}{n+2}$) and $x_k = (k+1)h$.

      ii. For what degree polynomials is this method exact?

   (b) Recall that the Chebyshev polynomials may be defined by $T_j(x) = \cos\left(j \cos^{-1} x\right)$ for $-1 \le x \le 1$.

      i. Show that these satisfy the recurrence relation $T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x)$ and use the recurrence relation to find $T_2(x)$ and $T_3(x)$ (having found T_0 and T_1 from the definition).

      ii. Show that the Chebyshev polynomials are orthogonal under the inner product defined with weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$.