Math 222: Diff. Eqns. Final Exam December 18, 2019

Name: (print) ____________________

Section Number: ____________________

Signature*: ____________________

*My signature affirms that this examination is completed in accordance with the NJIT Academic Integrity Code.

Instructions: Please complete the problems on the following pages in the space provided. If you need additional space to work, please use the back of the previous page. All work must be shown in order to receive full credit. Answers without explanation will receive *no* credit. The use of books, notes, calculators, smartphones, smartwatches, or any other external sources of information is not permitted during this examination.

The exam writer's promise: All the algebra works out neatly. If you ever find yourself with a "funny" number like $\sqrt{31}$ or $\frac{13}{47}$, that means you have made an algebra error.

Question	Points	Score
1	10	
2	6	
3	10	
4	14	
5	10	
6	12	
7	10	
8	10	
9	10	
10	8	
Total:	100	

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$

1. (10 points) Consider the ODE boundary value problem

$$y'' + 4y' + 5y = 0,$$
$$y(0) = 0, y(L) = 1,$$

where $L > 0$. For which values of L does this fail to have a solution?

2. (6 points) A certain mass-spring system is defined by

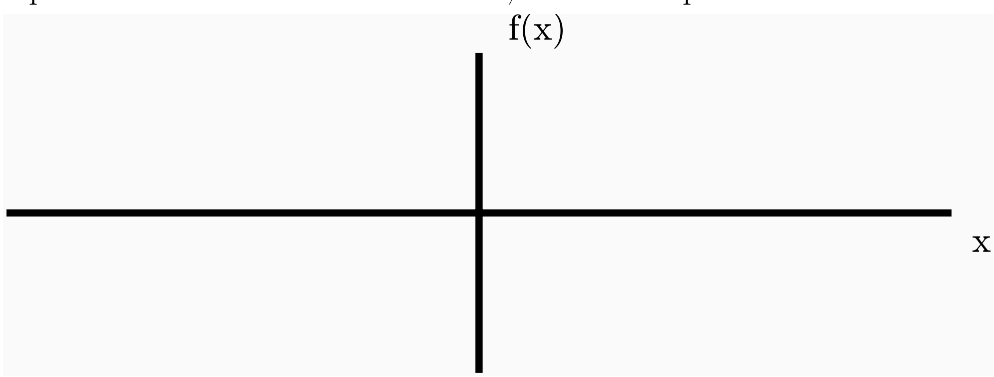
$$mu'' + \gamma u' + ku = 0$$

with $m = 3$ and $k = 12$. Find all physically-meaningful values of γ such that the system would be underdamped.

3. (10 points) Find the inverse Laplace transform of $F(s) = e^{-4s}\frac{s-1}{s^2+7s+10}$.

4. Let $f(x) = 4$ be defined for $0 \leq x \leq 2$.

(a) (4 points) Consider the *odd* periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods on the interval $-6 < x < 6$, of this odd periodic extension



(b) (2 points) Find a_{10}, the tenth cosine coefficient of the Fourier series of the periodic function you drew in part (a). (Hint: shortcuts are okay if you make a good argument.)

(c) (6 points) Solve the BVP for $u(x)$ using a Fourier *sine* series expansion.

$$u''(x) = f(x);\ 0 < x < \pi$$
$$u(0) = 0; u(\pi) = 0.$$

More specifically, solve analytically for the coefficients in the sine series of $u(x)$.

(d) (2 points) To what does the series converge at $x = -1$ and at $x = 2$?

5. (10 points) Find all the eigenvalues (positive, zero, and negative, if they exist) and eigenfunctions of

$$y''(x) + \lambda y(x) = 0, y(0) = 0, y'\left(\frac{\pi}{2}\right) = 0.$$

6. (a) (10 points) Solve the initial value problem:

$$y''(t) + 16y(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ 8 & \text{if } \pi \le t < 3\pi, \\ 0 & \text{if } 3\pi \le t; \end{cases} \quad y(0) = 0; y'(0) = 0,$$

   (b) (2 points) Plot your solution $y(t)$ and properly label the graph.

7. (a) (8 points) Find the solution to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   (b) (2 points) Is the origin a stable or unstable (No credit without explanation.)

8. (10 points) For each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = A\mathbf{x}$, where A is a nonsingular 2×2 matrix of real numbers. Based solely on the information given in each part, classify the point $(0, 0)$ **and** state whether it is stable or unstable.
   (a) One of the eigenvalues of A is $r = -1 + 8i$.
   (b) The characteristic equation of A is $r^2 - 7r + 12 = 0$.
   (c) The coefficient matrix is $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$.
   (d) The general solution is $x(t) = C_1 e^{(2-\sqrt{6})t}\binom{1}{1} + C_2 e^{(2-\sqrt{6})t}\binom{1}{-2}$.
   (e) Some, but not all, nonzero solutions become unbounded as $t \to \infty$.

9. (10 points) Find a fundamental solution set for the differential equation

$$y'' - 6y' + 9y = 0$$

and demonstrate that it is, indeed, a fundamental solution set.

10. (8 points) Solve the initial value problem

$$ty' + y = t,\ y(1) = 2.$$