1. Sketch the region bound by \( y = \ln(x) \), \( y = 0 \), and \( x = e \). Then set up the integral to find the volume of the figure formed by rotating this area about the line \( y = 3 \). Show a sketch of the region being revolved. DO NOT SOLVE THE INTEGRAL. (10 points)
2. Suppose that a spring has a natural length of 1 foot and that a force of 10 pounds is required to hold it compressed to a length of 6 inches. How much work is done in stretching the spring from its natural length to a total length of 2 feet? (10 points)

\[
\begin{align*}
\text{work: } W &= \int F \, dx \\
\text{linear spring: } F &= k(x - x_0) \\
W &= \int_{x_0}^{2 \text{ft}} k(x - x_0) \, dx \\
&= \int_{1 \text{ft}}^{2 \text{ft}} k(x - 1) \, dx \\
&= \left[ k \frac{x^2}{2} - kx_0x \right]_{x=1 \text{ft}}^{2 \text{ft}} \\
&= \left( 20 \text{lb/ft} \right) \left( 2 \text{ft}^2 \right) \\
W &= 40 \text{ ft-lb} \\
\end{align*}
\]

3. Suppose a line segment from the origin to the general point \((r, h)\) is rotated about the y-axis as shown, creating a right circular cone. Find the volume of this cone in terms of \(r\) and \(h\) by using either the method of cross sections or method of cylindrical shells (12 points)

\[
V = \int_{0}^{h} A(y) \, dy \\
A(y) = \pi R^2(y) \\
\text{where } R(y) \text{ is the cross-section's radius} \\
\]

**The cone has circular cross-sections which are perpendicular to the y-axis.**

\[
V = \int_{0}^{h} \pi \left( \frac{r}{h} y \right)^2 \, dy \\
V = \int_{0}^{h} \pi \left( \frac{r^2}{h^2} y^2 \right) \, dy \\
V = \left[ \pi \frac{r^2}{h^2} \frac{y^3}{3} \right]_{y=0}^{h} \\
V = \frac{\pi r^2 h}{3}
\]

\[
A(y) = \pi R^2(y) \\
R(y) = \frac{r}{h} y
\]

**This is a straight line.**

\[
b = 0, \quad m = \frac{r}{h} \\
\text{result } R(y) = \left( \frac{r}{h} \right) y
\]
4. Find the volume of the figure formed by rotating the area bound between \( y = x^2 \) and \( y = 2x \) around the y-axis. Use any method. (10 points)

**Shell method:**

- Axis of rotation: \( x = 0 \)
- Shell radius: \( r(x) = x - 0 \)
- Cylinder height: \( h(x) = 2x - x^2 \)
- Limits of integration: \( a = 0 \), \( b = 2 \)

Find curve intersections:
Solve \( h(x) = 0 \) will do.
\[ 2x - x^2 = 0 \]
\[ x(2 - x) = 0 \]
\[ x = 0 \quad \text{or} \quad x = 2 \]

\[ V = \int_0^2 2\pi h(x) r(x) \, dx \]
\[ = \int_0^2 (2\pi)(2x - x^2)(x) \, dx \]
\[ = \int_0^2 (2\pi)(2x^2 - x^3) \, dx \]
\[ = 2\pi \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 \]
\[ = 2\pi \left( \frac{2}{3}(2)^3 - \frac{1}{4}(2)^4 \right) \]
\[ = 2\pi \left( \frac{16}{3} - 4 \right) \rightarrow 4 = \frac{12}{3} \]
\[ V = \frac{8\pi}{3} \]
5. Find the volume of the figure formed by rotating the area bound by the x-axis and the cubic \( y = x^2 - x^3 \) about the y-axis. (12 points)

\[
\text{Shell method is the only choice}
\]

axis of rotation : \( x = 0 \)
shell radius : \( r(x) = x - 0 \)
shell height : \( h(x) = (x^2 - x^3) - 0 \)
limits of integ. : \( a = 0 \), \( b = 1 \)

\[
V = \int_0^1 (x^2 - x^3) \pi \, dx 
\]

\[
= 2\pi \int_0^1 (x^2 - x^3) \, dx 
\]

\[
= 2\pi \left[ \frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = 2\pi \left( \frac{1}{4} - \frac{1}{5} \right) = 2\pi \left( \frac{5}{20} - \frac{4}{20} \right) = \frac{\pi}{10} 
\]

6. Evaluate \( \int \frac{\cos(\sqrt{3}x)}{\sqrt{x}} \, dx \) (8 points):

- substitute: \( u = \sqrt{3}x \) \( \Rightarrow du = \frac{\sqrt{3}}{2\sqrt{x}} \, dx \)

then \( \frac{1}{\sqrt{x}} \, dx = \frac{2}{\sqrt{3}} \, du \)

- integral becomes:

\[
I = \frac{2}{\sqrt{3}} \int \cos(u) \, du 
\]

\[
I = \frac{2}{\sqrt{3}} \sin(u) + C
\]

\[
I = \frac{2}{\sqrt{3}} \sin(\sqrt{3}x) + C
\]
7. Find the arc length of the curve \( x = \frac{1}{6}y^3 + \frac{1}{2y} \) between \( y=1 \) and \( y=2 \) (12 points)

\[
S = \int_{y=1}^{y=2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

\[
\frac{dy}{dx} = \frac{3}{6}y^2 + \frac{-1}{2y^2} = \frac{1}{4}y^3 - \frac{1}{2}(\frac{y^2}{y}) + \frac{1}{4}y^4
\]

\[
\left( \frac{dy}{dx} \right)^2 = \frac{1}{4}y^6 + \frac{1}{2}y^4 + \frac{1}{4}y^8 - 2
\]

\[
S = \int_{y=1}^{y=2} \sqrt{1 + \frac{1}{4}(y^4 + y^{-4})} \, dy
\]

\[
S = \int_{y=1}^{y=2} \sqrt{1 - \frac{1}{2} + \frac{1}{4}y^4 + \frac{1}{4}y^{-4}} \, dy
\]

\[
S = \int_{y=1}^{y=2} \sqrt{\frac{1}{4}y^4 + \frac{1}{2}y^2 + \frac{1}{4}y^{-4}} \, dy
\]

\[
S = \int_{y=1}^{y=2} \frac{1}{4} \sqrt{y^4 + 2y^2 + y^{-4}} \, dy
\]

\[
S = \int_{y=1}^{y=2} \frac{1}{4} \sqrt{(y^2 + y^{-2})^2} \, dy
\]

Continuation:

\[
S = \int_{y=1}^{y=2} \frac{1}{2} (y^2 + y^{-2}) \, dy
\]

\[
S = \int_{y=1}^{y=2} \frac{1}{2} \left( \frac{1}{3}y^3 - \frac{1}{y} \right) \, dy
\]

\[
S = \int_{y=1}^{y=2} \left( \frac{8}{3} - \frac{1}{y} \right) - \left( \frac{1}{3} - \frac{1}{y} \right) \, dy
\]

\[
S = \int_{y=1}^{y=2} \left( \frac{7}{3} + \frac{1}{y} \right) \, dy
\]

\[
S = \frac{1}{2} \left( \frac{14}{6} + \frac{3}{6} \right)
\]

\[
S = \frac{1}{2} \left( \frac{17}{6} \right)
\]

\[
S = \frac{17}{12}
\]
8. Find the surface area of the paraboloid formed when the curve $x = y^2$, $0 \leq y \leq \sqrt{2}$ is revolved around the x-axis. (12 points)

Simplest way is to integrate the circular profile along the x-axis to generate the surface.

\[ A = \int_{0}^{\sqrt{2}} S(y) \, dy \]

$S(y)$: Circumference of the profile

\[ S(y) = 2\pi r = 2\pi \sqrt{x} \]

\[ A = \int_{0}^{\sqrt{2}} 2\pi \sqrt{x} \, dx = 2\pi \left[ \frac{2}{3} x^{3/2} \right]_{x=0}^{x=2} = \frac{4}{3} \pi \sqrt{2} - \frac{4}{3} \pi \cdot 2\sqrt{2} \]

\[ A = \frac{8\sqrt{2}}{3} \pi \]
9. Suppose that a cylindrical tank is buried upright underground on one of its circular bases. The tank has a height of 6 meters and a radius of 2 meters. Suppose the top of the tank is exactly 3 meters below the earth's surface. If the tank is half filled with water weighing approximately 10,000 N/m$^3$. Find the work needed to pump all the water in this tank to the ground's surface. (14 points)

\[
\begin{align*}
\text{Weight: } \quad mg &= 10000 \, \text{N/m}^3 \\
\text{Density: } \quad \rho g &= 10000 \, \text{N/m}^3
\end{align*}
\]

- $P_e$: potential energy
- $W$: work
- all potential energy is gravitational potential energy
  \[ P_e = -mgh \]
  where $h$ is height (altitude)
- define ground level to be $h = 0$
- then $P_e|_{\text{ground}} = 0$

Conservation of energy
\[ W - P_e_1 = -P_e_2 \]

- $P_e_2 = 0$ ground level
- water has different potential energy at different heights, so we have to "sum" (integrate)
\[ W = P_e_1 = \int g h \, dm \]
\[ = \int \rho g h A dh \]
\[ = -9m \]
\[ W = \int_{-9}^{-6} -\rho g h A dh \]

Recap: 
- \( h \) - vertical coordinate
- \( A \) - cross-sectional area of the cylinder
- \( \rho g = 10000 \text{ N/m}^3 \) - weight density of water given

\[ A = \pi r^2 \quad r = 2 \text{ m} \quad \text{given} \]

\[ W = \int_{-9}^{-6} -\rho g (\pi r^2) h dh \]

\[ = -\rho g \pi r^2 \left[ \frac{1}{2} h^2 \right]_{h=-9}^{-6} \]

\[ = \rho g \pi r^2 \left[ \frac{81 \text{ m}^2 - \frac{36 \pi^2}{4}}{2} \right] \]

\[ = \left( 10000 \frac{\text{ N}}{\text{ m}^3} \right) \pi (4 \text{ m}^2) \left[ \frac{72 \pi^2}{4} \text{ m}^2 \right] \]

\[ = (\pi)(1100000 \text{ Nm}) \]

\[ W = (\pi)(900000 \text{ Nm}) \]

or \( \pi 900 \text{ kJ} \)