Math 111 Exam #2
October 25, 2017

Time: 1 hour and 25 minutes
Instructions: Show all work for full credit.
No outside materials or calculators allowed.
Extra Space: Use the backs of each sheet for extra space. Clearly label when doing so.

Name: ______________________________

ID #: ______________________________

Instructor/Section: ____________________

"I pledge by my honor that I have abided by the NJIT Academic Integrity Code."

__________________________ (Signature)

Problem	Value	Score
1	15 pts.	
2	15 pts.	
3	12 pts.	
4	12 pts.	
5	12 pts.	
6	12 pts.	
7	12 pts.	
8	10 pts.	
TOTAL	100	

1. Consider the curve $y = f(x) := \frac{4}{x} + \frac{x}{2}$.

(a) Find the tangent line to the curve at (4,3). **(5 pts.)**

(b) Find the normal line to the curve at (4,3). **(5 pts.)**

(c) Find all points where the tangent line to the curve is horizontal. **(5 pts.)**

Solution 1:

(a) $y' = \frac{-4}{x^2} + \frac{1}{2} \Rightarrow y'(4) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$. Hence, the tangent line at (4,3) is

$$y - 3 = y'(4)(x-4) = \frac{1}{4}(x-4) \Leftrightarrow x - 4y + 8 = 0.$$

(b) The slope of the normal line is the negative reciprocal of the slope of the tangent line, so the normal line through (4,3) is

$$y - 3 = -\frac{1}{y'(4)}(x-4) = -4(x-4) \Leftrightarrow 4x + y + 13 = 0.$$

(c) The tangent line is horizontal when the derivative is zero, so from (a) we must solve $x^2 = 8$, which implies $x = \pm 2\sqrt{2}$. Hence, the points are $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$.

2. Let a body move along the *s*-axis with its position given as $s(t) = t^3 - 6t^2 + 9t$ in meters, with *t* in seconds. Find each of the following **(5 pts. each)**:

(a) The velocity *v* and acceleration *a*. (b) *a* when *v* = 0. (c) The speed when *a* = 0.

Solution 2:

(a) $v = \dfrac{ds}{dt} = 3t^2 - 12t + 9$ (m/sec) and $a = \dfrac{dv}{dt} = 6(t-2)$ (m/sec^2).

(b) From (a), *v* vanishes when $t^2 - 4t + 3 = 0 = (t-1)(t-3)$, and $a(1) = -6$, $a(3) = 6$ (m/sec^2).

(c) From (b), *a* = 0 when *t* = 2, so the speed is $|v(2)| = |3(2)^2 - 12(2) + 9| = |12 - 24 + 9| = |-3| = 3$ (m/sec).

3. Find the derivatives dy/dx of each of the following **(6 pts. each)**:

(a) $y = 2e^{-x} + xe^{3x}$ (b) $y = \dfrac{3x + \tan 2x}{x \sec x}$

Solution 3:

(a) We simply use the formula for the derivative of an exponential, the linearity rule, the addition rule and the chain rule to compute that

$$y' = -2e^{-x} + e^{3x} + 3xe^{3x} = -2e^{-x} + (1+3x)e^{3x}.$$

(b) Simply use the quotient rule plus the formulas for the derivatives of the trigonometric functions and the chain rule to obtain

$$\frac{dy}{dx} = \frac{(x\sec x)\left[3 + 2\sec^2 2x\right] - \left[\sec x + x\sec x \tan x\right](3x + \tan 2x)}{x^2 \sec^2 x}$$

$$= \frac{(x)\left[3 + 2\sec^2 2x\right] - \left[1 + x\tan x\right](3x + \tan 2x)}{x^2 \sec x} = \frac{2x\sec^2 2x - \tan 2x - x(3x + \tan 2x)\tan x}{x^2 \sec x}.$$

4. Find the derivatives dy/dx for each of the following **(6 pts. each)**:

(a) $(3xy+7)^2 = 6y$ (b) $y = (\sin x)^x$

Solution 4:

(a) Assuming that $y = y(x)$ is differentiable, implicit differentiation yields

$$2(3xy+7)(3y+3xy') = 6y' \Rightarrow y' = x(3xy+7)y' + y(3xy+7)$$

$$\Rightarrow y' = \frac{y(3xy+7)}{1-x(3xy+7)}.$$

(b) One can use logarithmic differentiation, or what is essentially equivalent, note that $y = e^{x\ln(\sin x)}$. Then using the usual formulas and the chain rule, one obtains

$$y' = (x\ln(\sin x))'e^{x\ln(\sin x)} = \left(\ln(\sin x) + x\frac{\cos x}{\sin x}\right)e^{x\ln(\sin x)} = (\ln(\sin x) + x\cot x)(\sin x)^x.$$

5. Show all work in obtaining the answers to each of the following **(6 pts. each)**:

(a) Suppose the function $y = f(x)$ is differentiable in an interval containing $x = 3$ and $f(3) = 8$ and $f'(3) = 2/5$, so that the function has a differentiable inverse f^{-1} in an interval containing $y = 8$. What is the derivative of the inverse function at $y = 8$?

(b) Find the derivative of $y = \sqrt{x}\arcsin\left(\sqrt{x}\right) = \sqrt{x}\sin^{-1}\left(\sqrt{x}\right)$.

Solution 5:

(a) Rather than trying to remember a formula, just recall that the inverse is obtained by solving $y = f(x)$ for x as a differentiable function of y, namely $x = x(y) = f^{-1}(y)$, which we know exists from the theorem in an interval containing $y = 8 = f(3)$ since $f'(3) \neq 0$. Using implicit differentiation on $y = f(x)$, we find that

$$1 = f(x(y))x'(y) = f(x(y))\left(df^{-1}/dy\right) \Rightarrow df^{-1}/dy = 1/df(f^{-1}(y)/dx.$$

Consequently, $\dfrac{df^{-1}}{dy}(8) = 1/\dfrac{df(3)}{dx} = 5/2.$

(b) Using the usual rules plus those for the derivatives of inverse functions, one obtains

$$y' = \frac{1}{2\sqrt{x}}\sin^{-1}(\sqrt{x}) + \sqrt{x}\,\frac{1}{2\sqrt{x}}\frac{1}{\sqrt{1-(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}}\sin^{-1}(\sqrt{x}) + \frac{1}{2\sqrt{1-x}}.$$

6. Find the derivatives of each of the following functions **(6 pts. each)**:

(a) $y = \cot^{-1}(1+3t)^{1/2} = \operatorname{arccot}(1+3t)^{1/2}$ (b) $y = \ln[(\sin\theta\cos\theta)^{1/2} / (1+2\ln\theta)]$.

Solution 6:

(a) Use the chain rule (twice) and the formula for the derivative of the inverse cot: Set $u = (1+3t)^{1/2}$ and compute that

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = -\frac{1}{1+u^2}\frac{1}{2}(1+3t)^{-1/2}(3) = -\frac{3}{2}\frac{(1+3t)^{-1/2}}{2+3t}.$$

(b) Here we use the chain rule again along with the other usual rules and formulas. First, we set $y = \ln u$, where

$$u = \frac{(\sin\theta\cos\theta)^{1/2}}{1+2\ln\theta}.$$

Hence, the chain rule and the formula for the derivative of the natural log yield

$$\frac{dy}{d\theta} = \frac{d}{du}\ln u\frac{du}{d\theta} = \frac{1+2\ln\theta}{(\sin\theta\cos\theta)^{1/2}}\frac{d}{d\theta}\left(\frac{(\sin\theta\cos\theta)^{1/2}}{1+2\ln\theta}\right),$$

whereupon, we use the quotient rule and some trigonometric identities to obtain

$$\frac{dy}{d\theta} = \frac{1+2\ln\theta}{(\sin\theta\cos\theta)^{1/2}}\frac{(1+2\ln\theta)(1/2)\left(\sin\theta\cos\theta\right)^{-1/2}(\cos^2\theta-\sin^2\theta)-(2/\theta)(\sin\theta\cos\theta)^{1/2}}{(1+2\ln\theta)^2}$$

$$= \frac{(1/2)(1+2\ln\theta)\cos(2\theta)-(1/\theta)\sin(2\theta)}{(\sin\theta\cos\theta)(1+2\ln\theta)} = \frac{(1+2\ln\theta)\theta\cos(2\theta)-2\sin(2\theta)}{\theta\sin(2\theta)(1+2\ln\theta)}.$$

7. A 13 ft. ladder leans against a vertical wall, with its base sliding along a horizontal floor. When the base is 5 ft. from the wall it is moving away from the wall at a rate of 2 ft./sec. along the floor. How fast is the ladder sliding down the wall at this instant? **(12 pts.)**

Solution 7:

Let the origin coincide with the point where the vertical wall intersects the horizontal floor, with y and x, respectively, be the distance from the floor and wall where the ladder touches the wall and floor. Then, it follows from the Pythagorean theorem that $x^2(t)+y^2(t)=(13)^2=169$. Note that when x = 5, y = 12. Differentiating this formula yields

$$2x\dot{x}+2y\dot{y}=0 \Rightarrow \dot{y}=-\frac{x}{y}\dot{x},$$

where the overdot, as usual, denotes differentiation with respect to t. At the instant in question, it is given that $\dot{x}=2, x=5, y=12,$ so the above equation shows that $\dot{y}=-5/6$ ft./sec.

8. Sand is being poured onto a conical pile at the rate of 3 cubic meters per second. The sand pile is a right circular cone (with circular base on the ground) of height *h* and diameter 2*r*, with the height equal to twice the base diameter. At what rate is the height increasing when the pile is 6 meters high? (Recall that the volume of the cone is $V = (\pi/3)r^2 h$) **(10 pts.)**

Solution 8:

We are given that $h = 4r$, so $V = \dfrac{\pi}{3\cdot 16}h^3$, which implies that

$$\frac{dV}{dt} = 3 = \frac{\pi}{16}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{48}{\pi h^2} = \frac{48}{36\pi} = \frac{4}{3\pi} \text{ m/sec.}$$