1. (a) Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear map. Show that $(\text{im}(T))^{\perp} = \ker(T')$ and $\text{im}(T) = (\ker(T'))^{\perp}$, where in the second equality we have used the canonical identification of an element in W with an element in its double dual W''.

   (b) Consider the vector space $V = \text{Span}\{\sin(2t), \cos(2t)\}$. Find the matrix associated with the linear operator $T : V \to V$ in the basis $\{\sin(2t), \cos(2t)\}$, where

$$T = \frac{d^2}{dt^2} + 5\frac{d}{dt} + 4I.$$

   (c) Let T and V be as in part (b). Is T invertible? If so, find an expression for T^{-1} and the associated matrix.

2. Let A be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix, $n \geq 2$. Show that $\det(EA) = \alpha \det(A)$, where

$$\alpha = \begin{cases} 1 & \text{if } E \text{ corresponds to a row-replacement} \\ -1 & \text{if } E \text{ corresponds to a row interchange} \\ r & \text{if } E \text{ corresponds to a scaling by } r \end{cases}.$$

3. Let * denote the adjoint of an operator. Let V and W be inner-product spaces and let $A : V \to W$ be an injective linear operator. For $b \in W$, let $\hat{x}$ be the least squares solution of $Ax = b$ given by

$$\hat{x} = \text{argmin}_{x \in V} \|Ax - b\|_W.$$

   (a) Show that $A\hat{x} = b_p$, where b_p is the orthogonal projection of b onto $\text{im}(A)$.

   (b) Show that A^*A is invertible and $\hat{x} = (A^*A)^{-1}A^*b$.

   (c) Use the QR decomposition to solve the least squares problem $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}.$$