Ph.D. Prelim: Exam. C
Probability Theory & Design of Experiments

May 23, 2018

Note for questions 1–3: Use the notation *a.e.* to denote almost everywhere, $\xrightarrow{\text{a.e.}}$, $\xrightarrow{\mathbb{P}}$, $\xrightarrow{\mathbb{D}}$ to denote convergence *a.e.*, in probability, and in distribution respectively, Φ to denote the standard normal cumulative distribution function, $S_n = \sum_{j=1}^{n} X_j$ or $S_n = \sum_{j=1}^{k_n} X_{nj}$, $I(A)$ to denote the indicator of event A, $P(A)$ for the probability of event A, and σ_{nj} for the standard deviation of X_{nj}, where $j = 1, \ldots k_n$ and $n = 1, 2, \ldots$. **For questions 4–6:** Assume that the error distributions in the models are independent and identically distributed $\mathrm{N}(0, \sigma^2)$.

1. This question has two **independent** parts, **(i)** and **(ii)** below.

(i) State and prove a theorem which gives a necessary and sufficient condition for convergence a.e. to 0 of a sequence $\{X_n, n \geq 1\}$ of random variables.

(ii) Let $\{E_n\}$ be arbitrary events satisfying

$$(a) \lim_n \mathbb{P}(E_n) = 0; \qquad (b) \sum_n \mathbb{P}(E_n E_{n+1}^c) < \infty.$$

Prove that $\mathbb{P}(\limsup_n E_n) = 0$.

$$\left[\text{HINT} : \bigcup_{n=m}^{M} E_n = \left(\bigcup_{n=m}^{M-1} E_n E_{n+1}^c \right) \cup E_M. \text{ If you choose to use the hint you need to prove it!} \right]$$

2. This question has two **independent** parts, **(i)** and **(ii)** below.

(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Prove that a sufficient condition for X to be a random variable is that for all x, $X^{-1}((-\infty, x]) \in \mathcal{F}$.

(ii) For any characteristic function f, prove that

$$|f(t) - f(t+h)|^2 \leq 2 \left\{ 1 - \mathbf{R} f(h) \right\},$$

where $\mathbf{R} f(h)$ is the real part of $f(h)$. Hence prove that f is uniformly continuous in R^1.

3. Let $X_1, X_2, \ldots$ be uncorrelated random variables such that $E(X_n) = 0$ and $E(X_n^2) \leq M$ for all n. Prove the following:

(i) $S_n / n^p \xrightarrow{\mathbb{P}} 0$ for $p > 1/2$.

(ii) $S_n^2 / n^{2p} \xrightarrow{\text{a.e.}} 0$ for $p > 3/4$.

(iii) $S_n / n^p \xrightarrow{\text{a.e.}} 0$ for $p > 3/4$.

4. Consider a single factor model (one-way analysis of variance model) for an unbalanced design with the usual assumptions of homoscedasticity, normality and independence. Find the estimators for the contrasts $\sum_{i=1}^{a} c_i \mu_i$ and $\sum_{i=1}^{a} d_i \mu_i$. Show that these estimators are independent of each other under the assumption of orthogonal contrasts.

5. Consider the Balanced Incomplete Block Design (BIBD).

(i) The treatment effect is estimated by $\hat{\tau}_i = \frac{kQ_i}{\lambda a}$, called the intrablock estimator. Derive the variance of $\hat{\tau}_i$, where $kQ_i = ky_{i.} - \sum_{j=1}^{b} n_{ij} y_{.j}$

(ii) Find a BIBD for an experiment with six blocks to compare four treatments in blocks of two runs.

6. State the two-factor analysis of variance full model. Write down the normal equations corresponding to all the parameters and derive their estimators.