Qualifying Exam

This is a closed-book exam. You must explain the details of your work for each problem for full credit!

1. (20pts)

(a) Characterize the type of singularity inside the unit disk of the function $f(z) := \dfrac{\cos(e^{-z})}{z^2}$ and evaluate

$$\int_{|z|=1} \frac{\cos(e^{-z})}{z^2} dz$$

(b) Use the Taylor series representation of $f(z) := \dfrac{1}{1-z}$ about $z_0 = 0$ for $|z| < 1$, find a series representation of $f(z)$ for $|z| > 1$.

2. (20pts) This problem aims to prove that *"A polynomial of degree m, $m \in \mathbb{Z}$, $m \geq 1$ has exactly m complex roots (counting multiplicity)"* using Liouville's Theorm and Rouché's Theorem. Let

$$P(z) := a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0,$$

$a_j \in \mathbb{C}$ for $j = 0, \ldots, m$; $a_m \neq 0$ be a polynomial of degree m.

(a) Using Liouville's Theorem, show that $P(z)$ has at least one complex root. From then, show that $P(z)$ has exactly m roots.

(b) Using Rouché's Theorem, show that for $R > 0$ large enough, $P(z)$ and the polynomial $Q(z) := z^m$ have the same number of roots inside the disk $D(0, R) := \{z \in \mathbb{C} : |z| < R\}$. From then, deduce that $P(z)$ has exactly m complex roots.

3. (10pts) Find the number of zeros of the following function in the first quadrant

$$f(z) = z^4 - 2z^2 - iz + i$$