Qualifying Exam

1. (a) Suppose that $f(z)$ is an analytic function in a domain $D \subset \mathbb{C}$. Prove that the complex conjugate $\overline{f}(z)$ is not analytic in D unless f is a constant function in D.

(b) Suppose that $f(z)$ is entire and there exists a constant $M \in \mathbb{R}$ such that $\text{Re}f(z) \leq M$ for all $z \in \mathbb{C}$. By applying Liouville's theorem to the function $g(z) = e^{f(z)}$, prove that $\overline{f}(z)$ is also entire.

2. (a) Suppose that $P_m(z)$ and $Q_n(z)$ are two polynomials of degrees m and n respectively and $n - m \geq 2$. Suppose further that the rational function $\dfrac{P_m(z)}{Q_n(z)}$ has no pole on the real axis. Prove that, the integral $\displaystyle\int_{-\infty}^{\infty} \frac{P_m(x)}{Q_n(x)} dx$ is $2\pi i$ times the sum of the residues of $\dfrac{P_m(z)}{Q_n(z)}$ at its poles in the upper half plane.

(b) Apply the result in part (a), evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} dx.$$

3. Let $f(z) = e^{1/z}$.

(a) Prove that $z = 0$ is an essential singularity of function f.

(b) Find the Laurent series of f at $z = 0$.

(c) Note that the coefficient of the term z^k in the Laurent series in part (b) is given by

$$a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{1/\eta}}{\eta^{k+1}} d\eta,$$

where γ is the unit circle $\{z : |z| = 1\}$ followed in the positive (anticlockwise) direction. Use the parameterization $z(\theta) = e^{i\theta}$, $-\pi \leq \theta \leq \pi$ for all $z \in \gamma$, show that

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\cos\theta} \cos(\sin\theta + k\theta) d\theta.$$

(d) Using results in parts (b) and (c), show that for all $n \in \mathbb{Z}$, $n \geq 0$,

$$\frac{1}{\pi} \int_{0}^{\pi} e^{\cos\theta} \cos(\sin\theta - n\theta) d\theta = \frac{1}{n!}.$$