Problems for Complex Analysis Qualifying Exam, Spring 2022

Problem 1 By integrating a suitable complex analytic function $f(z)$ around a suitable closed contour, and applying the Residue Theorem, evaluate

$$I = \int_0^\infty \frac{1-\cos x}{x^2(1+x^4)} dx.$$

Problem 2 (Rouché's theorem)

(a) How many zeros does $F(z) = z^5 + z^4 + z^3 + z + 5$ have inside the unit disk?

(b) Let $f(z) = z^n$ and $g(z) = \sum_{k=0}^{n-1} a_k z^k$, $a_k \in \mathbb{C}$. Show that, for $|z|$ sufficiently large, $|f(z)| > |g(z)|$. Deduce that any polynomial of degree n having complex coefficients (with nonzero leading coefficient) has exactly n roots in $\mathbb{C}$, counted according to multiplicity. Can you provide a bound for the size of the largest root in terms of the a_k?

Problem 3 (Transform methods). For the following problem you may assume that the Fourier Transform of $F(x)$ and its inverse are given by

$$\hat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ikx} dx, \qquad F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(k) e^{-ikx} dk;$$

that the Laplace Transform of $G(t)$ and its inverse are given by

$$\bar{G}(s) = \int_0^\infty G(t) e^{-st} dt, \qquad G(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\sigma - iR}^{\sigma + iR} \bar{G}(s) e^{st} ds,$$

provided $\sigma \in \mathbb{R}$ is chosen such that $\bar{G}(s)$ is complex analytic for $\Re(s) > \sigma$; and that, where the transforms exist, second derivatives transform as

$$\widehat{F_{xx}}(k) = -k^2 \hat{F}(k) \quad \text{(Fourier Transform)},$$

$$\overline{G_{tt}}(s) = s^2 \bar{G}(s) - sG(0) - G'(0) \quad \text{(Laplace Transform)}.$$

Consider the wave equation on an unbounded domain, $-\infty < x < \infty$ and for positive times $t \geq 0$:

$$u_{tt} = u_{xx}, \quad |u| \to 0 \text{ as } |x| \to \infty, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

(a) By taking the Laplace Transform followed by the Fourier Transform, show that

$$\hat{\bar{u}}(k,s) = \frac{s\hat{f}(k) + \hat{g}(k)}{s^2 + k^2} \qquad (*).$$

(b) By inverting the Laplace Transform in $(*)$, show that

$$\hat{u}(k,t) = \hat{f}(k) \cos kt + \frac{\hat{g}(k)}{k} \sin kt \qquad (\dagger).$$

(You may assume that the relevant contributions on the semicircular portion of the Bromwich contour go to zero as $R \to \infty$.)

(c) In the special case where $g(x) = 0$, apply the Fourier inversion theorem to $(\dagger)$ to show that

$$u(x,t) = \frac{1}{2}\big(f(x-t) + f(x+t)\big).$$

Hint: Use the complex exponential representation for $\cos kt$ in (b).