Ph.D. Qualifying Exam. C
Probability Theory and Design of Experiment

Aug 18, 2020

Note for questions 1–3: Use the notation *a.e.* to denote almost everywhere, $\xrightarrow{\text{a.e.}}$, $\xrightarrow{P}$, $\xrightarrow{\mathcal{D}}$ to denote convergence *a.e.*, in probability, and in distribution respectively, Φ to denote the standard normal cumulative distribution function.

1. This question has two parts, **(i)** and **(ii)** below.

(i) Let Y be a random variable such that $E(|Y|^k) < \infty$ for some $k \geq 1$. Give a complete proof (*any result that is exploited must be proved as well*) that

$$\lim_{n\to\infty} n^k \mathbb{P}(|Y| > n) = 0.$$

(ii) Using the result in part 1(i), show that if $Y_1, \ldots, Y_n$ are independent and identically distributed random variables with finite second moment, then

$$\frac{1}{\sqrt{n}} \max_{1\leq j\leq n} |Y_j| \xrightarrow{P} 0.$$

2. This question has independent parts **(i)** and **(ii)** below.

(i) Suppose that $\{Z_n, n \geq 1\}$ is a sequence of random variables. If for each $\epsilon > 0$, $\sum_{n=1}^{\infty} \mathbb{P}(|Z_n| > \epsilon) < \infty$ then $Z_n \xrightarrow{\text{a.e.}} 0$. [NOTE: Any theorem that is exploited must be proved as well.]

(ii) State and prove any five important properties of the characteristic function associated with a probability measure μ.

3. Suppose X_n are independent and $\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = n^{-\alpha}/4$ and $\mathbb{P}(X_n = 0) = 1 - n^{-\alpha}/2$, with $0 < \alpha < 3$. Derive an appropriate Central Limit Theorem (CLT) for $S_n = X_1 + \ldots + X_n$, giving a precise range for α over which the stated CLT would hold. [NOTE: $\sum_{j=1}^{n} j^{2-\alpha}/2$ increases to ∞ at the order of $n^{3-\alpha}$.]

4. Consider the assumptions, blocking effects $\beta_j, j = 1, ..., b$, are uncorrelated with zero means and same variance σ_β^2 in a balanced incomplete block design (BIBD). Also assume that the $\{\beta_j, j = 1, ..., b\}$ is independent of $\{\epsilon_{ij}, all\ i, j\}$, and ϵ_{ij}'s are independent identically distributed (i.i.d.), with mean 0, variance σ^2. Derive the interblock estimator $\tilde{\tau}_i$ and show that it is an unbiased estimator of τ_i.

5. Consider the randomized complete block design described by

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij},$$

$i = 1, 2, \ldots, a, j = 1, 2, \ldots, b$. **(i)** Use the reduction in the sum of squares for this full model to derive an expression for the sum of square error (SSE). **(ii)** Use the expression for SSE in this problem part (i) to describe the method for imputing the missing data y_{ij}, i and j fixed. Derive a closed form expression for the missing value.

6. Consider the single random factor model described by

$$y_{ij} = \mu + \tau_i + \epsilon_{ij},$$

$i = 1, 2, \ldots, a; j = 1, 2, \ldots, n$. Here, $\{\tau_i\}$ is i.i.d. normal with mean zero and variance σ_τ^2, $\{\epsilon_{ij}\}$ is i.i.d. normal with mean zero and variance σ^2, and $\{\tau_i\}$ is independent of $\{\epsilon_{ij}\}$. Derive the test that checks the significance of this model. Give explanations for how the test distribution under the null hypothesis is derived.