Doctoral Qualifying Exam
New Jersey Institute of Technology
Department of Mathematical Sciences

Statistics Part B: Real Analysis and Statistical Inference — August 2020

The first three questions are about Real Analysis and the next three questions are about Statistical Inference.

1. Let (Ω, Σ, μ) be a σ-finite measure space and let $f : \Omega \to \mathbb{R}$ be a non-negative Σ-measurable function. For every set $A \in \Sigma$ define $\nu(A) := \int_A f(x)\, d\mu(x)$.

   (a) State the defining properties of μ as a measure.

   (b) Prove that ν is also a measure on Σ.

   (c) Prove that if $g : \Omega \to \mathbb{R}$ is another measurable non-negative function, then

   $$\int_\Omega g(x)\, d\nu(x) = \int_\Omega f(x)g(x)\, d\mu(x).$$

   Hint: Use the level set definition of the Lebesgue integral and Fubini's theorem.

2. Let $c > 0$ and let $g : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ := (0, \infty) \subset \mathbb{R}$, be $\mathcal{L}^1$-measurable.

   (a) Use Hölder's inequality to prove that

   $$\int_{\mathbb{R}^+} \frac{|g(t)|^3}{(c^2 + t^2)^{1/4}}\, dt \leq C\|g\|^3_{L^4(\mathbb{R}^+)},$$

   where $C > 0$ is a constant that depends only on c. Give an explicit example of such a constant.

   (b) The result in part (a) implies that the integral in the left-hand side is finite, if $g \in L^4(\mathbb{R}^+)$. Does this statement still hold if $g \in L^2(\mathbb{R}^+)$? If $g \in L^\infty(\mathbb{R}^+)$? Either prove these statements or provide a counterexample.

   (c) Is the left-hand side of the inequality in part (a) finite for $g(t) = \frac{\sin t}{t^{5/4}}$? What about the right-hand side?

3. Show that $|x| = xH(x) - xH(-x)$ where $H(x)$ indicates the Heaviside function. Then use this relation to compute the Fourier transform of $|x|$.

   Hint: You may need to use $\mathcal{F}H(\omega) = \frac{\delta}{2} + \frac{1}{2\pi i} vp(1/x)$ where $\mathcal{F}$ indicates the Fourier transform and vp the principal value.

4. Let $X_1, \ldots X_n$ be independent random variables with densities

   $$f_{X_i}(x;\theta) = \frac{1}{2i\theta}, -i(\theta - 2) \leq x \leq i(\theta + 2),$$

   $f_{X_i}(x;\theta) = 0$, elsewhere, $0 < \theta < \infty, i = 1, 2, ..., n$, respectively.

   (a) Find the maximum likelihood estimator (MLE) of θ.

(b) Also, find the MLE of $Var|\ \frac{X_i}{i} - 2\ |$, where $|\ .\ |$ is the absolute value function.

(c) State and prove at least one property of the MLE of θ.

5. Suppose that the m-dimensional random vector X_n has a $N_m(\mu, \Sigma_n)$ distribution, i.e. multivariate normal with mean μ and variance-covariance matrix Σ_n, which is positive definite, $n = 1, 2, \ldots$. Also suppose that the matrix Σ_n converges to the matrix Σ, which is positive definite. Find the limiting distribution of $Y_n = (X_n - \mu)'\Sigma^{-1}(X_n - \mu)$ as n goes to ∞.

6. Let $X_i, i = 1, 2, ..., n$ be a random sample from $N(\theta_1, \theta_2)$, where $\theta_2 > 0$ is variance of the normal random variable and θ_1, is the unspecified mean. Derive the likelihood ratio test of size α for testing $H_0 : \theta_2 = \theta_2'$ versus $H_1 : \theta_2 \neq \theta_2'$, where $\theta_2' > 0$ is a specified value.