Doctoral Qualifying Exam
New Jersey Institute of Technology
Department of Mathematical Sciences

Applied Math Part C: Linear Algebra and Numerical Methods August 2020

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. Given the matrix

$$A = \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix}$$

and the vector

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

compute

$$\lim_{n\to\infty} \frac{\|A^{n+2}x\|}{\|A^n x\|} \text{ and } \lim_{n\to\infty} \frac{\|A^{-(n+2)}x\|}{\|A^{-n}x\|} .$$

2. Let $*$ denote the conjugate transpose. Hermitian matrices satisfy that $A^* = A$, unitary matrices satisfy that $A^*A = I$, and normal matrices satisfy that $A^*A = AA^*$. Prove the following

   (a) The eigenvalues of a Hermitian matrix are real.

   (b) The eigenvectors of a Hermitian matrix for distinct eigenvalues are orthogonal.

   (c) The eigenvalues of a unitary matrix have modulus 1.

   (d) For u a unit vector, the matrix $A = I - 2uu^*$ is unitary.

   (e) Let u and A be as in part (d), with $u \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$ for some $n > 1$. The eigenvalues of A are -1 and 1.

3. Let $*$ denote the adjoint of an operator. Normal operators satisfy that $T^*T = TT^*$.

   (a) If W is the kernel (nullspace) of a normal transformation T, then $T^*(W) \subset W$.

   (b) Let $T = \frac{d}{dx} : V \to V$ be the differentiation operator acting on the space V of degree-1 polynomials in the basis

   $$\{x - 1, x + 1\} .$$

   Find the matrices associated with operators T and T^* in this basis, and show that the matrix associated with T is not normal.

   (c) With T and V as in part (b), what are the eigenvalues of T acting on V?

   (d) With T and V as in part (b), is there any basis for V in which T is normal? Explain.

4. (a) Prove that the fixed point iteration

$$x_{i+1} = x_i^2 - 2$$

converges to 2 only if $x_n = 2$ for some finite $n \in \mathbb{Z}$.

(b) Consider instead the fixed point iteration

$$x_{i+1} = c(x_i^2 - 2).$$

Which values of $c \in \mathbb{R}$ ensure that this converges to a fixed point given a sufficiently good initial guess x_0?

5. Let $h > 0$ and suppose $f \in C^3$ on $[-h, h]$.

(a) Derive a quadrature formula for approximating the integral

$$\int_0^h f(x)\, dx$$

that makes use of the values of $f(-h)$, $f(0)$, and $f(h)$.

(b) Derive an error estimate for your quadrature formula.
Hint: Make use of the error formula for *nth* degree polynomial interpolation

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi(x))(x - x_0) \cdots (x - x_n).$$

6. Consider the numerical solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0$$

using the implicit two-step method

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h}{2} \left(f(t_{n+1}, y_{n+1}) - f(t_{n-1}, y_{n-1}) \right)$$

with the exact starting values y_0, y_1.

(a) Determine the leading term in the local truncation error. What is the order of the method?

(b) Is this a convergent method? Explain.