Doctoral Qualifying Exam
New Jersey Institute of Technology
Department of Mathematical Sciences

Applied Math Part B: Real and Complex Analysis — August 2020

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Let (Ω, Σ, μ) be a σ-finite measure space and let $f : \Omega \to \mathbb{R}$ be a non-negative Σ-measurable function. For every set $A \in \Sigma$ define $\nu(A) := \int_A f(x)\, d\mu(x)$.

   (a) State the defining properties of μ as a measure.

   (b) Prove that ν is also a measure on Σ.

   (c) Prove that if $g : \Omega \to \mathbb{R}$ is another measurable non-negative function, then

   $$\int_\Omega g(x)\, d\nu(x) = \int_\Omega f(x) g(x)\, d\mu(x).$$

   Hint: Use the level set definition of the Lebesgue integral and Fubini's theorem.

2. Let $c > 0$ and let $g : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ := (0, \infty) \subset \mathbb{R}$, be $\mathcal{L}^1$-measurable.

   (a) Use Hölder's inequality to prove that

   $$\int_{\mathbb{R}^+} \frac{|g(t)|^3}{(c^2 + t^2)^{1/4}}\, dt \le C \|g\|^3_{L^4(\mathbb{R}^+)},$$

   where $C > 0$ is a constant that depends only on c. Give an explicit example of such a constant.

   (b) The result in part (a) implies that the integral in the left-hand side is finite, if $g \in L^4(\mathbb{R}^+)$. Does this statement still hold if $g \in L^2(\mathbb{R}^+)$? If $g \in L^\infty(\mathbb{R}^+)$? Either prove these statements or provide a counterexample.

   (c) Is the left-hand side of the inequality in part (a) finite for $g(t) = \frac{\sin t}{t^{5/4}}$? What about the right-hand side?

3. Show that $|x| = xH(x) - xH(-x)$ where $H(x)$ indicates the Heaviside function. Then use this relation to compute the Fourier transform of $|x|$.

   Hint: You may need to use $\mathcal{F}H(\omega) = \frac{\delta}{2} + \frac{1}{2\pi i} vp(1/x)$ where $\mathcal{F}$ indicates the Fourier transform and vp the principal value.

4. Consider the following problems for the complex plane $\mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\}$:

   (a) Sketch the subset $S := \{z \in \mathbb{C} : |z - \bar{z}| > 2\}$ and determine whether or not it is (i) open, (ii) connected or (iii) bounded.

   (b) Use the Cauchy–Riemann equations to show that the function

   $$f(z) := \log|z| + i \arg z = \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

   is analytic for $0 < |z|$ and $|\arg z| < \pi$ and determine its derivative.

5. (a) Consider the function $F(z) := \frac{1}{z(z-1)}$, which is meromorphic on the entire complex plane. Show that $\int_C F(z)dz = 0$, for C the circle defined by $|z| = 2$ with the usual counterclockwise orientation, in two ways: (i) direct use of the Residue Theorem; and (ii) using the Zero-Pole Theorem, which requires that you find a function f satisfying $f'/f = F$.

   (b) Prove that if g is entire and satisfies $|g(z)| \leq M|z|^n$ for some positive constant M and $|z|$ sufficiently large, then it must be a polynomial of degree $d \leq n$.

   (c) What can one say about an entire function h that is zero for all points of the following sequences: (i) $\{z_n = 1/n : n \in \mathbb{N}\}$ or (ii) $\{z_n = n : n \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the natural numbers (or positive integers). For (ii) also consider the Casorati–Weierstrass Theorem.

6. Consider the following problems related to the maximum and argument principles.

   (a) Determine the maximum and minimum moduli of e^{z^2} on the closed unit disk $D := \{z \in \mathbb{C} : |z| \leq 1\}$.

   (b) Prove that the number of zeros of $f(z) := 3z^n - e^{z^2}$ in D is equal (counting multiplicity) to n for any positive integer n.