Suggested Analysis Qualiifer Questions May 2022

1. Let $\{f_n\}$ be a sequence of integrable functions on a measure space $(X, \mathbf{X}, \mu)$ such that
$$\sum_{n=1}^{\infty} \int_X |f_n| \; d\mu < \infty$$
Show that the series $\sum_{n=1}^{\infty} f_n$ converges absolutely almost everywhere in X, the sum is integrable on X and that
$$\int_X (\sum_{n=1}^{\infty} f_n) \; d\mu = \sum_{n=1}^{\infty} \int_X f_n \; d\mu$$

2. Let $\{f_n\}$ be a sequence of measureable functions on a measure space $(X, \mathbf{X}, \mu)$ where $\mu(X) < \infty$. Show that if f_n converges almost everywhere to a function f that is finite almost everywhere, then f_n also convergences in measure. Show that this result may fail if $\mu(X) = \infty$.

3. Let f be a 2π-periodic function and consider the Fourier series generated by f given by $a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.
   (a) Use integration by parts, to show that if $f \in C^1$ that there exists $M_1 > 0$ such that $|a_n| < M_1/n$ and $|b_n| < M_1/n$ for all n.
   (b) Now suppose $f \in C^k$. Generalize the above result to obtain bounds for $|a_n|$ and $|b_n|$ that hold for all n.
   (c) Use the Riemann-Lebesgue Lemma to show that if $f \in C^k$, then $\lim_{n\to\infty} n^k |a_n| = 0$ and $\lim_{n\to\infty} n^k |b_n| = 0$.