

DOCTORAL QUALIFYING EXAM  
New Jersey Institute of Technology  
Department of Mathematical Sciences

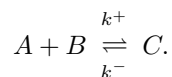
Applied Math Part A: Applied Mathematics

January 2021

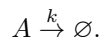
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**The first three questions are about Math 613 and the next three questions are about Math 651.**

1. Consider the following reversible reaction, modeled as a **deterministic** well-mixed mass-action system:



- (a) Write down the ordinary differential equations for the concentration variables  $A(t)$ ,  $B(t)$  and  $C(t)$ .
  - (b) Find two conservation laws for this system, and use them to eliminate variables  $A(t)$  and  $B(t)$ , leaving a single equation for the concentration of substance  $C$ . Assume that  $A(0) = B(0) = 0$ , and  $C(0) = C_0 \neq 0$ .
  - (c) Non-dimensionalize the remaining equation. Hint: the result should be of form  $c' = p(1 - c)^2 - c$ , where  $p = \text{const}$  (or something similar to this form).
  - (d) Make a qualitative plot of the non-dimensionalized concentration  $c$  as a function of time, in the special case  $C_0 = k^-/k^+$ . You don't have to determine the exact solution (although this wouldn't be hard).
2. Consider a discrete-state **continuous-time** Markov death process (e.g. stochastic cell death, molecular degradation, or nuclear decay):



- (a) Write down the Markov diagram and the Chemical Master Equations for this process, and derive the PDE for the probability-generating function,  $F(z, t) = \sum_{n=0}^{\infty} p_n(t) z^n$ .
  - (b) Find the initial condition for the probability-generating function,  $F_0(z) \equiv F(z, 0)$ , assuming that at  $t = 0$  the particle number distribution is Poisson:  $p_n(0) = e^{-\lambda} \lambda^n / n!$  (where  $\lambda = \text{const} > 0$ ,  $n = 0, 1, 2, \dots$ ).
  - (c) Find  $F(z, t)$  with the given initial condition, using the method of characteristics. Make a rough plot of several characteristics.
  - (d) Use your solution for  $F(z, t)$  to find the average number of particles, the variance, and the probability of having exactly one particle remaining, as functions of time.
3. Derive the diffusion equation for a thin, long tube with sealed side surface, and with cross-section area  $A(x)$  and diffusion coefficient  $D(x)$  smoothly varying along the tube's length. Hint: derive the conservation law relating molecular flux  $J(x)$  to the concentration  $u(x)$  for the case  $A(x) \neq \text{const}$  by considering rate of change of molecule number in a section of the tube. Then, combine this conservation law with the Fick's law of diffusion to arrive at the final PDE.
4. (a) Find the general solution to the ODE  $x^2 y'' - xy' + y = x^2$ . Simplify your answer.

- (b) Find the solution to  $y'' + 3y' + 2y = e^{-x}$  with  $y(0) = 0$  and  $y'(0) = 0$ .
- (c) Solve the PDE  $xu_x + yu_y = 1 + y^2$  with  $u(x, 1) = x$  by using the method of characteristics. Explain your steps.
5. Consider the system  $x_t = xy - 1$  and  $y_t = x - y^3$ .
- (a) Is the system Hamiltonian? Why or why not? If yes, find the Hamiltonian.
- (b) Find the fixed points and determine their nature. Indicate any local stable or unstable directions.
- (c) Find the nullclines.
- (d) Draw the phase portrait where the fixed points, trajectories, nullclines, stable and unstable directions are clearly indicated.
6. Suppose that for  $0 < x < 1$ ,

$$u_t = u_{xx} + \sin(\pi x)$$

subject to boundary conditions  $u(0, t) = 0$ ,  $u(1, t) = 1$  and initial condition  $u(x, 0) = x + \sin(3\pi x)$ . Find the exact form of  $u(x, t)$  and explain all the steps.