Doctoral Qualifying Exam
New Jersey Institute of Technology
Department of Mathematical Sciences

Applied Math Part A: Applied Mathematics January 2021

The first three questions are about Math 613 and the next three questions are about Math 651.

1. Consider the following reversible reaction, modeled as a **deterministic** well-mixed mass-action system:

$$A + B \underset{k^-}{\overset{k^+}{\rightleftharpoons}} C.$$

   (a) Write down the ordinary differential equations for the concentration variables $A(t)$, $B(t)$ and $C(t)$.

   (b) Find two conservation laws for this system, and use them to eliminate variables $A(t)$ and $B(t)$, leaving a single equation for the concentration of substance C. Assume that $A(0) = B(0) = 0$, and $C(0) = C_0 \neq 0$.

   (c) Non-dimensionalize the remaining equation. Hint: the result should be of form $c' = p(1-c)^2 - c$, where $p = const$ (or something similar to this form).

   (d) Make a qualitative plot of the non-dimensionalized concentration c as a function of time, in the special case $C_0 = k^-/k^+$. You don't have to determine the exact solution (although this wouldn't be hard).

2. Consider a discrete-state **continuous-time** Markov death process (e.g. stochastic cell death, molecular degradation, or nuclear decay):

$$A \xrightarrow{k} \varnothing.$$

   (a) Write down the Markov diagram and the Chemical Master Equations for this process, and derive the PDE for the probability-generating function, $F(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$.

   (b) Find the initial condition for the probability-generating function, $F_0(z) \equiv F(z,0)$, assuming that at $t = 0$ the particle number distribution is Poisson: $p_n(0) = e^{-\lambda}\, \lambda^n/n!$ (where $\lambda = const > 0$, $n = 0, 1, 2, ...$).

   (c) Find $F(z,t)$ with the given initial condition, using the method of characteristics. Make a rough plot of several characteristics.

   (d) Use your solution for $F(z,t)$ to find the average number of particles, the variance, and the probability of having exactly one particle remaining, as functions of time.

3. Derive the diffusion equation for a thin, long tube with sealed side surface, and with cross-section area $A(x)$ and diffusion coefficient $D(x)$ smoothly varying along the tube's length. Hint: derive the conservation law relating molecular flux $J(x)$ to the concentration $u(x)$ for the case $A(x) \neq const$ by considering rate of change of molecule number in a section of the tube. Then, combine this conservation law with the Fick's law of diffusion to arrive at the final PDE.

4. (a) Find the general solution to the ODE $x^2y'' - xy' + y = x^2$. Simplify your answer.

(b) Find the solution to $y'' + 3y' + 2y = e^{-x}$ with $y(0) = 0$ and $y'(0) = 0$.

(c) Solve the PDE $xu_x + yu_y = 1 + y^2$ with $u(x, 1) = x$ by using the method of characteristics. Explain your steps.

5. Consider the system $x_t = xy - 1$ and $y_t = x - y^3$.

   (a) Is the system Hamiltonian? Why or why not? If yes, find the Hamiltonian.

   (b) Find the fixed points and determine their nature. Indicate any local stable or unstable directions.

   (c) Find the nullclines.

   (d) Draw the phase portrait where the fixed points, trajectories, nullclines, stable and unstable directions are clearly indicated.

6. Suppose that for $0 < x < 1$,

$$u_t = u_{xx} + \sin(\pi x)$$

subject to boundary conditions $u(0, t) = 0$, $u(1, t) = 1$ and initial condition $u(x, 0) = x + \sin(3\pi x)$. Find the exact form of $u(x, t)$ and explain all the steps.