Suggested Analysis Qualiifer Questions May 2023

1. (25 points) Let $(X, \mathcal{M}, \mu)$ be the Lebesgue measure space with $X = \mathbf{R}$, $\mathcal{M}$ = Lebesgue measurable sets and μ the Lebesgue measure. Let E be a measurable set.

   (a) Let $\{f_n\}$ be a sequence of nonnegative measurable functions such that $f_n \to f$ almost everywhere on E. Suppose $f_n(x) \leq f(x)$ for all $x \in E$. Show

   $$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$$

   Hint: Use Fatou's Lemma for part of your arguement.

   (b) Let f_n be a sequence of measurable functions that converges uniformly to f. Provide an example where f_n does not converge in $L_1(X)$. State and prove conditions under which uniform convergence of f_n implies convergence in $L_p(X)$ to f for $1 \leq p < \infty$.

2. (20 points) Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $b = (b_1, \ldots, b_n)$ and A be an $n \times n$ real valued matrix $A = (a_{ij})_{1\leq i,j\leq n}$. Define the mapping $T : \mathbf{R}^n \to \mathbf{R}^n$ in the following way

   $$y = T(x) = Ax + b$$

   which can be written component wise in the form

   $$y_i = \sum_{j=1}^{n} a_{ij} x_j + b_i.$$

   (a) Consider $\rho(x, x^*) = \max_{1\leq i\leq n} |x_i - x_i^*|$ for $x, x^* \in \mathbf{R}^n$. Prove that ρ is a metric on $\mathbf{R}^n$.

   (b) Using the metric ρ, derive conditions so that T is a contraction mapping.

3. (15 points) The goal of this problem is to prove that term wise integration of a Fourier series holds. You **cannot** invoke that result; you

must explicitly show all calculations. Let f be continuous on $[0, 2\pi]$ and 2π periodic with the Fourier series generated by f given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

Let

$$F(x) = \int_0^x [f(t) - \frac{a_0}{2}]\ dt.$$

Note that F is absolutely continuous and $F' = f$ almost everywhere. Explain why the Fourier series generated by F converges everywhere to F, and can be written

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos nx + B_n \sin nx]$$

For $n \geq 1$, show that $A_n = -b_n/n$ and $B_n = a_n/n$. Finally use the above condition to find the value of A_0 in terms of a_n and b_n.