Suggested Analysis Qualiifer Questions May 2023

1. (a) Let $X = \mathbf{N}$, $\mathbf{X}$ be the σ-algebra of all subsets of X, and μ the counting measure. Let f be a non-negative measureable function. Show that

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} f(n)$$

   (b) Let $(X, \mathbf{X}, \mu)$ be a finite measure space. If f is a measureable function let

$$r(f) = \int_X \frac{|f|}{1+|f|} \, d\mu.$$

   Show that a sequence of measureable functions $\{f_n\}$ converges in measure if and only if $r(f_n - f) \to 0$.

2. Let $f : [-\pi, \pi] \to \mathbf{R}$ be differentiable on $(-\pi, \pi)$ such that $f(\pi) = f(-\pi)$. Let the Fourier seris of f be given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

   (a) Use integration by parts to compute the Fourier coeffcientcs for $f'(x)$.

   (b) Prove that the Fourier series converges pointwise to $f(x)$. Is the convergence also uniform?

   (c) Consider the function $f(x) = x \sin \frac{1}{x} \, dx$ with $f(0) = 0$. Show that it fails to satisfy the assumptions of the problem at $x = 0$, but by appling Dini's Theorem, convergence of the Fourier Series can still be obtained. As a reminder, Dini's Theorem states that if $g(t) = (f(x+t) + f(x-t))/2$ on $t \in [0, \delta]$, if $s(x) = g(0+)$ exists and if $(g(t) - s(x))/t \in L[0, \delta]$ for some $\delta < \pi$, then the Fourier series generated by f converges to $s(x)$.

3. Let $f : S \to S$ be a function from a complete metric space (S, ρ) into itself. Assume there is a sequence $\{\alpha_n\}$ which converges to 0 such that $\rho(f^n(x), f^n(y)) \leq \alpha_n \rho(x, y)$ for all $n \geq 1$ and all $x, y \in S$, where f^n is the nth iterate of f; that is $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$. Prove the f has a unique fixed point.