Doctoral Qualifying Exam B
Applied Math: Real & Complex Analysis

August, 2019

1. (a) State the definition of a measure.
   (b) Prove that if $A \subset B$, then $\mu(A) \le \mu(B)$.
   (c) Prove that if $A_1 \subset A_2 \subset A_3 \subset \ldots$, then

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

   (d) Prove that if $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\mu(A_1) < \infty$, then

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right).$$

2. For $a > 0$, let

$$h(a) := \int_{\mathbb{R}} \frac{dx}{a^2 + x^2}.$$

   (a) Use the level set definition of the Lebesgue integral to compute $h(a)$ and compare your answer with the result from classical calculus.
   (b) Use Lebesgue dominated convergence theorem to pass to the limit $a \to \infty$ under the integral sign to compute $\lim_{a\to\infty} h(a)$.
   (c) Now pass to the limit $a \to \infty$ in the result of part (a) and show that it agrees with the result obtained in part (b). Can the approach of part (b) be applied to compute $\lim_{a\to\infty} ah(a)$?

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

   (a) Does this function belong to $C(\mathbb{R}^3)$? Does it have a compact support? Does it belong to $C_c(\mathbb{R}^3)$? Does it belong to $L^p(\mathbb{R}^3)$ for any $1 \le p \le \infty$?
   (b) Compute the Fourier transform $\widehat{f}$ of f.
   (c) Show that $\widehat{f} \in L^2(\mathbb{R}^3)$.

4. (a) Determine the number of zeros of $e^z - 4z^2 - 1 = 0$ in $|z| < 1$.
(b) Using the Argument Principle, determine the number of zeroes located inside the first quadrant of the function $f(z) = z^5 + 1$.

5. Evaluate

$$(a) \int_0^{\pi/4} \sin^4 \theta d\theta \quad \text{and, (b)} \quad \int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2}$$

6. Consider the function

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + ... + \frac{1}{n!z^n}$$

(a) What is counted by the integral

$$\frac{1}{2\pi i} \oint_{|z|=r} \frac{f'(z)}{f(z)} dz?$$

(b) What is the value of the integral for large n and fixed r?
(c) What does this tell you about zeros of $f(z)$ for large n?