Ph.D. Prelim: Exam. C
Probability Theory and Design of Experiments

August 15, 2019

Note for questions 1–3: Use the notation *a.e.* to denote almost everywhere, $\xrightarrow{\text{a.e.}}, \xrightarrow{\mathbb{P}}, \xrightarrow{\mathcal{D}}$ to denote convergence *a.e.*, in probability, and in distribution respectively, Φ to denote the standard normal cumulative distribution function, $S_n = \sum_{j=1}^{n} X_j$ or $S_n = \sum_{j=1}^{k_n} X_{nj}$, $I(A)$ to denote the indicator of event A, $\mathbb{P}(A)$ for the probability of event A, and σ_{nj} for the standard deviation of X_{nj}, where $j = 1, \dots k_n$ and $n = 1, 2, \dots$.

1. This question has two parts, **(i)** and **(ii)** below.

(i) State and prove a necessary and sufficient condition for a sequence $\{X_n\}$ to converge to X *a.e.*

(ii) If $\sum_n \mathbb{P}(|X_n| > n) < \infty$, then prove that $\limsup_n |X_n|/n \le 1$ *a.e.*

2. This question has two parts, **(i)** and **(ii)** below.

(i) Let f denote the probability density function and φ denote the characteristic function. Prove that if $\varphi \in L^1$, then $f \in L^1$ and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \varphi(t) dt.$$

(ii) State and prove the "uniqueness theorem" for the "determining" measure μ or the distribution function F.

3. This question has two parts, **(i)** and **(ii)** below.

(i) State precisely the Liapounov and Lindeberg-Feller central limit theorems.

(ii) For the independent random variables $\{X_{nj}, 1 \le j \le n, n \ge 1\}$ with

$$\mathbb{P}(X_{nj} = 1) = \frac{1}{n - j + 1} = 1 - \mathbb{P}(X_{nj} = 0),$$

derive an appropriate central limit theorem.

4. Derive the normal equations for the Balanced Incomplete Block Design (BIBD), simplifying the equations as much as possible. Use it to derive the estimator of the ith treatment effect $\hat{\tau}_i$.

5. Consider the two-factor fixed effects model with replication. Assume that $\{\epsilon_{ijk}, i = 1, ..., a, j = 1, ..., b, k = 1, ..., n\}$ are independent identically distributed normal random variables, with mean 0, variance σ^2. Derive the test to evaluate the model is significant.

6. Consider a factorial experiment with two factors (A and B) with interaction and n replicates full model. Now suppose that to run this experiment a particular raw material is required. This raw material is available in batches that are not large enough to allow all abn treatment combinations to be run from the same batch. However, if a batch contains enough material for ab observations, then an alternative design is to run each of the n replicates using a separate batch of raw material. Consequently, the batches of raw material represent a randomization restriction or a block, and a single replicate of a complete factorial experiment is run within each block. (a) Give the effects model for this new design. (b) Describe the complete analysis of variance (ANOVA) table, no need to give the formulas for Sum of Squares or Mean Squares.