Doctoral Qualifying Exam C
Applied Math: Numerical Methods

August, 2019

1. Parts (a) and (b) of this question are unrelated.
   - (a) Find the natural cubic spline $s(x)$ interpolant of the given data: $(-1, 1), (0, -1)$ and $(2, 1)$. (Recall a natural cubic spline satisifes $s''(-1) = s''(1) = 0$).
   - (b) What are sufficient conditions on a function $f : \mathbb{R} \to \mathbb{R}$ that guarantee the existence of a unique fixed point $x_* = f(x_*)$ in the interval $[a, b]$? Under these conditions, prove that the iteration $x_{n+1} = f(x_n)$ converges at least linearly to x_* for any starting point $x_0 \in [a, b]$.

2. Consider the method

$$y_{n+1} = y_{n-1} + \frac{h}{8}(5f_{n+1} + 6f_n + 5f_{n-1})$$

for approximating the solution of the ODE

$$y'(t) = f(t, y).$$

Here, $f_n = f(t_n, y_n)$ and it can be assumed that the given starting values y_0 and y_1 are exact.

   - (a) Determine the leading order term in the local truncation error. What is the order of the method?
   - (b) Find the characteristic polynomial for this method and its roots r_1, r_2.
   - (c) Define $A-$stability of a method. What conditions must r_1, r_2 satisfy in order for the method to be A-stable? (Set up, but do not carry out the computations.)

3. For this question you may use the identities

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b), \qquad 2 \cos a \cos b = \cos(a - b) + \cos(a + b).$$

   - (a) Show that the Chebyshev polynomials

$$T_n(x) = \cos(n \arccos(x))$$

   are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$.
   - (b) Prove the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 2.$$

(c) Derive the 3-point Chebyshev-Gauss quadrature formula for approximating

$$I(f) = \int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}} dx \simeq \sum_{i=1}^{3} w_i f(x_i).$$

Hint: Notice that $\frac{x^n}{\sqrt{1-x^2}}$ is an odd function if n is odd and an even function if n is even.

(d) Use the 3-point formula to approximate $I(f)$ for $f(x) = e^{-2x^2}$.