Doctoral Qualifying Exam
Department of Mathematical Sciences
New Jersey Institute of Technology

Applied Math Part B: Real and Complex Analysis — August 2018

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Let $f \in L^1(\mathbb{R})$ and let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) := \int_{\mathbb{R}} \frac{f(y)}{1+(x-y)^2}\, dy.$$

   (a) Prove that $g(x)$ is differentiable at $x = 0$ and that

$$g'(0) = \int_{\mathbb{R}} \frac{2yf(y)}{(1+y^2)^2}\, dy.$$

   (*Hint:* apply the definition of the derivative and Lebesgue dominated convergence theorem.)

   (b) Verify the formula in (a) for $f(x) = \chi_{(0,R)}(x)$, where χ is the characteristic function and $R > 0$ is fixed, by explicitly computing $g(x)$ and differentiating it at $x = 0$.

   (c) Is the right-hand side of the formula defining $g(x)$ well-defined as a Lebesgue integral for $f \in C_c^\infty(\mathbb{R})$? For $f \in C^\infty(\mathbb{R})$? Either prove your statement or provide a counter-example.

2. Let $F : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ be defined as

$$F(u) := -\int_{\mathbb{R}}\int_{\mathbb{R}} u(x)u(y)\,|x-y|\,dx\,dy.$$

   (a) Show that if $v \in C_c^\infty(\mathbb{R})$, then

$$F(v') = 2\int_{\mathbb{R}} |v(x)|^2 dx,$$

   where $v' = dv/dx$. (*Hint:* integrate by parts.)

   (b) If $\widehat{u}$ is the Fourier transform of $u \in C_c^\infty(\mathbb{R})$ defined by $\widehat{u}(k) := \int_{\mathbb{R}} e^{-2\pi ikx}u(x)\,dx$, and u satisfies $\int_{\mathbb{R}} u(x)dx = 0$, show that

$$F(u) = \int_{\mathbb{R}} \frac{|\widehat{u}(k)|^2}{2\pi^2k^2} dk.$$

   (*Hint:* use the formula for the Fourier transform of the derivative.)

   (c) Show that the formula in part (b) may no longer be true, if $\int_{\mathbb{R}} u(x)dx \neq 0$.

3. Let $\langle\cdot,\cdot\rangle : C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R}) \to \mathbb{R}$ be defined as

$$\langle f, g\rangle := -\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\,|x-y|\,dx\,dy.$$

(a) State the defining properties of an inner product.

(b) Show that the function $\langle\cdot,\cdot\rangle$ above defines an inner product over functions in $C_c^\infty(\mathbb{R})$ that integrate to zero over $\mathbb{R}$.

(c) Show that $\langle\cdot,\cdot\rangle$ may be continuously extended to functions in $L^2(\mathbb{R})$ that integrate to zero over $\mathbb{R}$ and vanish outside a compact set, i.e., if $f_n, g_m \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} g_m(x)dx = 0$ and $\text{supp}(f_n) \cup \text{supp}(g_m) \subset K$ for some compact set $K \subset \mathbb{R}$ and all $n, m \in \mathbb{N}$, and

$$f_n \overset{L^2(\mathbb{R})}{\longrightarrow} f, \qquad g_m \overset{L^2(\mathbb{R})}{\longrightarrow} g \qquad \text{as } n, m \to \infty,$$

then

$$\langle f, g\rangle := \lim_{n,m\to\infty} \langle f_n, g_m\rangle = -\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\,|x-y|\,dx\,dy,$$

and $\langle\cdot,\cdot\rangle$ still defines an inner product.

4. (a) Prove the isolated zero theorem; namely, if f is a nonconstant analytic function in a domain $D \subset \mathbb{C}$, then if $f(z_0) = 0$ for $z_0 \in D$, there exists an $\epsilon > 0$ such that f is not zero for any point of the punctured neighborhood $\mathcal{P}_\epsilon(z_0) := \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$. *Hint: A good starting point is the series representation*

$$f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

(b) Consider the sequence of polynomials

$$\{p_n(z)\} := \left\{z\textstyle\prod_{k=2}^{n+1}(1 - k^2z^2) = z(1 - 2^2z^2)(1 - 3^2z^2)\cdots(1 - (n+1)^2z^2)\right\}.$$

Show using (a) that if this sequence converges uniformly to a function p on the unit disk $B := \{z \in \mathbb{C} : |z| \le 1\}$, then p must be identically zero. *Hint: Recall the theorem about the nature of such a limit, and note the resulting zeros of the function.*

5. Let the complex-valued function φ be analytic on the upper half-plane $H_+ := \{z = x + iy \in \mathbb{C} : y \ge 0\}$. Use residue theory and a semicircular contour and a semicircular contour indented at z to show that if $\alpha < 0$ and $|\varphi(z)| \le M\,|z|^\alpha$ for all $z \in H_+$, where M is a positive constant, then one has the formula (related to the Hilbert transform)

$$\int_{-\infty}^{\infty} \frac{\varphi(\zeta)d\zeta}{\zeta - z} = \begin{cases} 2\pi i\varphi(z), & z = x + iy \text{ with } y > 0 \\ \pi i\varphi(z), & z = x + iy \text{ with } y = 0 \end{cases}$$

6. Consider the following problems related to the principle of the argument and its associated results such as Rouché's theorem and the zero-pole theorem and its variants.

(a) Show that if f is analytic on the unit disk $B := \{z \in \mathbb{C} : |z| \le 1\}$ and $|f(z)| < 1$ for every z on the unit circle $\partial B := \{z \in \mathbb{C} : |z| = 1\}$, f has a unique fixed point in the interior of the disk; i.e., $f(z_*) = z_*$ for precisely one point with $|z_*| < 1$.

(b) If f is the same as in (a), what can be said about solutions of $z^m = f(z)$ in B, for any integer $m \geq 2$?

(c) Let $\Phi_n(z) := z\prod_{k=2}^{n+1}(1 - k^2z^2)$ for any positive integer n. Show, preferably without any lengthy computation, that

$$\frac{1}{2\pi i}\int_{\partial B} \frac{\Phi_n'(z)dz}{\Phi_n(z)} = 1 + 2n.$$

(d) Similarly, show that

$$\frac{1}{2\pi i}\int_{\partial B} \frac{z\Phi_n'(z)dz}{\Phi_n(z)} = 0.$$