Department of Mathematical Sciences
New Jersey Institute of Technology

Part A: Applied Mathematics — Doctoral Qualifying Exam, August 2018

The first three questions are based on Math 613 and the next three questions are about Math 651.

1. (a) Consider the transport of thermal energy in a one-dimensional region, with constant cross-sectional area A and no heat source, that occurs through the combination of advection and diffusion. Derive the differential equation for the conservation of thermal energy in the region:

$$\rho c(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}) = k\frac{\partial^2 T}{\partial x^2},$$

where T is the temperature, ρ the density, c the heat capacity, k (constant) coefficient of diffusion, and u the (constant) flow velocity in the x-direction.
(b) What are the units of k? Non-dimensionalize this equation and discuss any dimensionless parameters that appear.

2. Consider the boundary value problem

$$\epsilon y'' + y' = 2x, \quad 0 < x < 1, \quad 0 < \epsilon \ll 1$$
$$y(0) = 1, \quad y(1) = 1$$

(a) Find the exact solution.
(b) Use singular perturbation method to find outer, inner, and composite expansion to the solution of the boundary value problem.

3. (a) Formulate a model for the conservation of cars in a flow of traffic. Let ρ be the traffic density and let q be the flux of traffic (i.e. number of cars per hour passing a fixed location). Recall that the traffic flux q equals the density of cars ρ times their velocity u. Suppose that $u = u_{max}(1 - \rho/\rho_{max})$, where u_{max} and ρ_{max} are maximum velocity and maximum car density, respectively.
(b) Determine the traffic density (for $t > 0$) if the initial traffic is:

$$\rho(x,0) = \begin{cases} \rho_{max} & x < 0\,, \\ \rho_{max}(1 - x/2) & 0 < x < 1\,, \\ \rho_{max}/2 & x > 1\,. \end{cases}$$

Sketch the characteristics in the xt-plane and the traffic density as a function of x at $t = 0$ and a later time.

4. (a) Find the general solution for the differential equation

$$x^2y'' - (2x + 2x^2)y' + (x^2 + 2x + 2)y = 0.$$

(b) Find the first three terms of the Frobenius solution around $x = 0$ for

$$xy'' + e^x y = 0.$$

Use this Frobenius solution to show that the other solution has a logarithmic term.

5. Consider a system of two linear differential equations for $x(t)$ and $y(t)$:

$$\dot{x} = x + y + a, \quad \dot{y} = -2x - 2y + b.$$

a and b are constant coefficients.

(a) For the homogeneous case ($a = b = 0$), find all fixed points of the system. Sketch the phase portrait for the system. Determine the stability of the fixed point(s).

(b) For the non-homogeneous case where both a and b are non-zero, find the solvability condition. For a and b satisfying the solvability condition that you find, determine the general solution.

6. Solve the differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

for $r \in [0, 1]$ and $t > 0$, with the initial condition $u(r, t = 0) = g(r)$ and boundary condition $u(r = 1, t) = 0$.