DOCTORAL QUALIFYING EXAM
Department of Mathematical Sciences
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Applied Math Part C: Linear Algebra and Numerical Methods AUGUST 2018

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. (a) Let A be any $n \times n$ matrix. Determine whether matrices A and $A+I$ are similar.

   (b) Let A be an invertible $n \times n$ matrix. Then show that $cond_2(A) = 1$ if and only if A is a multiple of a unitary matrix U, where $cond_2$ is the condition number with respect to the 2-norm.

2. (a) Let $\mathbf{x} = c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n$, where $\mathbf{u}_i$, $i = 1, \ldots, n$, form a basis for the vector space in which $\mathbf{x}$ lies. Prove that $c_1, \ldots, c_n$ are unique.

   (b) Let L be the set of vectors $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T$ in $\mathbb{R}^4$ for which $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for L. What is the dimension of L?

3. Let:

$$y_1' = 5y_1 - 6y_2$$
$$y_2' = 3y_1 - 4y_2.$$

   Find the solution to the system that satisfies $y_1(0) = 4$, $y_2(0) = 1$.

4. (a) Will Newton's method converge quadratically to a root of $g(x) = x^2$? Fully justify your answer.

   (b) Suppose $f(x)$ has three continuous derivatives and suppose $q(x)$ is a polynomial of degree two that interpolates f at the nodes x_0, x_1, x_2, with $x_0 < x_1 < x_2$. Let $h = \max\{(x_1 - x_0), (x_2 - x_1)\}$ and $K = \max_{x \in [x_0, x_2]} |f'''(x)|$. Show that

$$\max_{x \in [x_0, x_2]} |f''(x) - q''(x)| = Ch^\alpha,$$

   by finding constants α and $C = C(K) > 0$. (Hint: Use the Newton form of the interpolating polynomial to show that there is a number $\eta \in [x_0, x_2]$ with $f''(\eta) - q''(\eta) = 0$. Then, integrate f'''.)

5. Consider the following scheme for solving the differential equation $y' = f(t, y)$:

$$y_{n+1} = y_n + \frac{h}{2}(y_n' + y_{n+1}') + \frac{h^2}{12}(y_n'' - y_{n+1}''),$$

   where $y_n' = f(t_n, y_n)$ and $y_n'' = f_t(t_n, y_n) + f_y(t_n, y_n) f(t_n, y_n)$.

   (a) Find the order of the scheme.

(b) Show that the region of stability contains the negative real axis $\mathbb{R}^-$.

6. Suppose that $w(x)$ is a weight function on $[a,b]$ and $\{p_k(x)\}_{k=0}^n$ is a family of orthogonal polynomials with respect to the inner product

$$(f,g) = \int_a^b f(x)g(x)w(x)dx.$$

Let $x_j, j = 1,\ldots,n$ be the roots of the polynomial $p_n(x)$, (note that orthogonality implies p_n has n simple roots contained in the interval (a,b)). Consider the quadrature formula given by

$$I(f) = \int_a^b f(x)w(x)dx \simeq Q(f) = \sum_{j=1}^n w_j f(x_j),$$

where

$$w_j = \int_a^b l_j(x)w(x)dx, \qquad l_j(x) = \prod_{i=1,\ldots n, i\neq j} \frac{(x-x_j)}{(x_i-x_j)}.$$

(a) Show that the degree of precision of the quadrature formula is less than or equal to $2n-1$.

(b) Show that the degree of precision of the quadrature formula can be no more than $2n-1$.