Doctoral Qualifying Exam
Department of Mathematical Sciences
New Jersey Institute of Technology

Applied Math Part B: Real Analysis August 2017

The following three questions are about Real Analysis.

1. Use the definition of the Lebesgue integral in terms of level sets to compute
$$\int_{\mathbb{R}^2} e^{-|x|^2}\,dx. \tag{1}$$
Then verify your answer, using the usual integration in polar coordinates.

2. Let $p > 1$ and let $F_p : L^p(\mathbb{R}) \to \mathbb{R}$ be defined by
$$F_p(f) := \sup_{\|g\|_q \le 1} \int_{\mathbb{R}} f(x)g(x)\,dx, \tag{2}$$
where the supremum is taken over $g \in L^q(\mathbb{R})$ with $q = p/(p-1)$.

   (a) State the definition of a norm on a vector space X.

   (b) Show that F_p defines a norm on $L^p(\mathbb{R})$.

   (c) Show that the supremum in (2) is attained and give an explicit expression for $F_p(f)$ in terms of f alone.

3. Let $a > 0$ and $b > 0$ and let $f(x) := ae^{-a^2x^2}$ and $g(x) := be^{-b^2x^2}$, with $x \in \mathbb{R}$.

   (a) Use the definition of the convolution integral to calculate $h = f * g$. Verify that $f * g = g * f$.

   (b) Now use Fourier transform to compute h and verify that the answer agrees with the one in part (a).

   (c) What happens with the answer in part (a) when $a \to \infty$? Can the assumptions of the Lebesgue dominated convergence theorem be satisfied in the definition of $h(x)$ as $a \to \infty$?