Doctoral Qualifying Exam

Department of Mathematical Sciences
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Applied Math Part C: Numerical Methods — August 2017

The following three questions are about Numerical Methods.

1. Let the function $f(x)$ satisfy $f(\alpha) = 0$ with $\alpha \neq 0$ and $0 < f'(x) < 1$. For each of the following fixed point iterations, find the range of values of the constant C (if any) for which the iteration is guaranteed to converge to α given a sufficiently good initial guess x_0.

   (a) $x_{n+1} = x_n + Cx_n^2 f(x_n)$

   (b) $x_{n+1} = x_n + Cf(x_n)^2$

2. (a) Let $f(x)$ be a smooth function and consider the integral

   $$\int_0^h x^{1/3} f(x)\, dx.$$

   Find the order of accuracy of the approximation

   $$I(h) \equiv h^{4/3} \left(\frac{9}{28} f(0) + \frac{3}{7} f(h) \right).$$

   (b) Let $f(x)$ be a smooth function and consider the integral

   $$\int_{-1}^1 x^{1/3} f(x)\, dx.$$

   Find the two-point Gaussian quadrature formula of the form

   $$\tilde{I} \equiv C_1 f(x_1) + C_2 f(x_2)$$

   that maximizes the precision of the approximation.

3. Consider the Runge-Kutta method

   $$y_{n+1} = y_n + \frac{1}{2} h \left[f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n)) \right]$$

   for approximating the solution of the ODE

   $$y'(t) = f(t, y).$$

   (a) Show that the method is second-order.

   (b) Find the real part of the region of absolute stability. (Hint: Consider $f(t, y) = \lambda y$ with $\lambda < 0$).