DOCTORAL QUALIFYING EXAM
Department of Mathematical Sciences
New Jersey Institute of Technology

Applied Math Part B: Real and Complex Analysis — MAY 2018

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Let $a > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be $\mathcal{L}^1$-measurable and not identically zero.

   (a) Use Hölder's inequality to prove that
   $$\int_{\mathbb{R}} \frac{|f(x)|^3}{(a^2 + x^2)^{1/4}}\, dx \le C\|f\|^3_{L^4(\mathbb{R})},$$
   where $C > 0$ is a constant that depends only on a. Give an explicit example of such a constant.

   (b) The result in part (a) implies that the integral in the left-hand side is finite, if $f \in L^4(\mathbb{R})$. Does this statement still hold, if $f \in L^2(\mathbb{R})$? If $f \in L^\infty(\mathbb{R})$? Either prove these statements or provide a counterexample.

   (c) Is the left-hand side of the inequality in part (a) finite for $f(x) = \frac{\sin x}{x^{5/4}}$? What about the right-hand side?

2. Consider the map $T : C([0,1]) \to C([0,1])$ defined by
   $$Tf(t) := \frac{1}{4}\int_0^t \frac{1 - f(\tau)}{\sqrt{t-\tau}}\, d\tau \qquad t \in [0,1].$$

   (a) Show that this definition is consistent, i.e., that if $f \in C([0,1])$, then also $Tf \in C([0,1])$ (<u>Hint:</u> a change of variables $s = (t-\tau)/t$ in the integral and Lebesgue dominated convergence theorem may be useful).

   (b) Show that T is a contraction map in $C([0,1])$ equipped with the usual norm $\|f\| = \sup_{0\le t\le 1} |f(t)|$, and use it to conclude that the map has a unique fixed point.

   (c) Write down the first three successive approximations to the fixed point, starting with $f = 0$ as an initial guess.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by
   $$f(x) = \begin{cases} 0, & x \le -1, \\ x, & -1 < x < 1, \\ 0, & x \ge 1, \end{cases} \qquad g(x) = \begin{cases} \frac{\sin(2\pi x) - 2\pi x\cos(2\pi x)}{2\pi^2 x^2}, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

   (a) Can the formula
   $$\widehat{f}(k) = \int_{\mathbb{R}} e^{-2\pi i k x} f(x) dx$$
   be used to compute the Fourier transform $\widehat{f}$ of f? If so, find $\widehat{f}(k)$.

(b) Show that the formula in part (a), in which the integral is understood in the sense of Lebesgue, may not be used to compute the Fourier transform $\widehat{g}$ of g.

(c) Show that the Fourier transform $\widehat{g}$ of g exists as an element in $L^2(\mathbb{R})$ and find it.

4. Use complex analysis to prove the fundamental theorem of algebra; namely, every complex polynomial $P(z)$ of degree $n > 0$ has precisely n complex zeros (roots) counting multiplicities.

5. (a) Derive the following formula for the residue at a pole z_0 of order m of a meromorphic function f on a complex domain D:

$$\operatorname{Res}\{f, z_0\} = \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} \left[(z-z_0)^m f(z)\right]_{|z=z_0}.$$

(b) Use (a) and a contour integral to find the Fourier transform

$$F(f)(\xi) = \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx$$

of the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) := 1/(x^2+1)^2$.

6. Consider the cubic equation $z^3+3z-w = 0$, where z and w are complex. The following concern zeros of this equation; that is, $z = z(w)$ such that $z^3(w) + 3z(w) - w = 0$. The equation can also be written in the form $f(z) - w = 0$, where $f(z) := z^3 + 3z$.

(a) Show that f has only the single zero at the origin in $B_{\sqrt{3}}(0) := \{z \in \mathbb{C} : |z| < \sqrt{3}\}$.

(b) Show that the minimum of $|f(z)|$ on the unit circle $C_1 := \{z \in \mathbb{C} : |z| = 1\}$ is equal to 2.

(b) Use Rouché's theorem and a variant of the zero-pole theorem for meromorphic functions
$$\left(\int_C g(z)(f'(z)/f(z))dz = 2\pi i\left(\sum_{f(z_k)=0} g(z_k) - \sum_{f(\zeta_k)=\infty} g(\zeta_k)\right)\right)$$
to find the power series representation of the zero $z = z(w)$ of the cubic equation lying in the unit disk $B_1(0)$.