Department of Mathematical Sciences
New Jersey Institute of Technology

Part A: Applied Mathematics — Doctoral Qualifying Exam, May 2018

The first three questions are based on Math 613 and the next three questions are about Math 651.

1. Consider the heat flow in a one-dimensional rod, $0 \le x \le 2$, of constant cross-sectional area, composed of two different materials, of equal size, with a different physical properties, in a perfect thermal contact; i.e. the temperature is continuous and no heat loss at where two materials join. Consider a heat source for $0 < x < 1$. A constant temperature of T_0 is imposed at $x = 0$ and a constant temperature of T_1 is imposed at $x = 2$ with an initial temperature distribution $f(x)$.

   (a) State the most general set of partial differential equations along with the corresponding boundary conditions to describe the problem.

   (b) Consider that for $0 < x < 1$, all the material properties are constant and 1 with the source term also to be 1, whereas for $1 < x < 2$, all the material properties are constant and 2. Determine the equilibrium temperature with $T_0 = T_1 = 0$.

2. Consider the initial value problem

$$\frac{d^2y}{dt^2} + 2\epsilon\frac{dy}{dt} + y = 0, \quad t \ge 0, \quad y(0) = 0, \quad \frac{dy(0)}{dt} = 1,$$

   where $< \epsilon \ll 1$.
   (a) Find the exact solution.
   (b) Use the straightforward asymptotic expansion to approximate the solution, keeping all $O(\epsilon)$ terms. Discuss the validity of the approximated solution.
   (c) Use a two-time scale asymptotic expansion to find a better approximation of the solution. Hint: choose the two time variables based on the exact solution.

3. Consider the arclength functional

$$J(y(x)) = \int_a^b \sqrt{1 + (y'(x))^2}dx,$$

   where $y(x)$ is twice continuously differentiable for $x \in [a, b]$, and $y(a) = y_a$ and $y(b) = y_b$. Find the unique extremal of the functional J.

4. Consider the hypergeometric equation

$$x(1-x)y'' + [\gamma - (1 + \alpha + \beta)x]\, y' - \alpha\beta y = 0,$$

   where α, β, and γ are constants.

   (a) Assume that $1 - \gamma$ is not a positive integer, show that, in the neighborhood of $x = 0$, one solution is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \cdots .$$

Do you expect the radius of convergence of this series to be smaller, equal to or larger than 1?

(b) Show that a second solution for $0 < x < 1$ is

$$y_2(x) = x^{1-\gamma}\left[1 + \frac{(\alpha-\gamma_1)(\beta-\gamma+1)}{(2-\gamma)1!}x + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{(2-\gamma)(3-\gamma)2!}x^2 + \cdots\right].$$

(c) By making the change of variable $\xi = 1/x$ to rewrite the hypergeometric equation in the new variable ξ, show that the point at infinity is a regular singular point and that the roots of the indicial equation are α and β.

5. Consider the system

$$\dot{x} = y^2 + xy - 2, \quad \dot{y} = x - y.$$

(a) Find all fixed points of the system.

(b) Determine the type and stability of each fixed point.

(c) Sketch the phase portrait for this system.

6. Consider a fourth-order linear differential operator for $x \in (0, 1)$

$$L = \frac{d^4}{dx^4}.$$

(a) Show that L is self-adjoint if u and v are any two functions satisfying the boundary conditions

$$\phi(0) = 0, \quad \phi(1) = 0, \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d^2\phi}{dx^2}(1) = 0.$$

(b) For the eigenvalue problem using the boundary conditions in part (a):

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. Further, show that eigenvalues $\lambda \leq 0$. Is $\lambda = 0$ an eigenvalue? If yes, find the corresponding eigenfunction.

Useful Formulas

* Euler-Lagrange equation:

If $y_1, \cdots, y_n$ provide a local minimum for the functional

$$J(y_1, \cdots, y_n) = \int_a^b L(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

where $y_i \in C^2[a,b]$ and $y_i(a) = \alpha_i$ and $y_i(b) = \beta_i$, $i = 1, \cdots, n$, then y_i must satisfy the Euler-Lagrange system of n ordinary differential equations

$$L_{y_i} - \frac{dL_{y_i'}}{dx} = 0, \quad i = 1, \cdots, n, \quad x \in [a,b].$$