Doctoral Qualifying Exam
Department of Mathematical Sciences
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Applied Math Part C: Linear Algebra and Numerical Methods May 2018

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. (a) Let A be a real 3×3 symmetric matrix with eigenvalues 0, 3, and 5 and corresponding eigenvectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. (i) Find a basis for the nullspace of A and a basis for the column space of A. (ii) If possible, find the solution(s) of $A\mathbf{x} = \mathbf{u}$. (iii) Is A invertible?
Solve the problem for the general case. Do not assume a specific matrix.

   (b) i. Let A and B be similar matrices. Show that they have the same determinant.
   ii. Let A and B be similar matrices. Show that they have the same characteristic equations and, thus, eigenvalues. **Do not use (ii) to prove (i).**

2. (a) Use the Gauss-Seidel method to find a solution to the linear system $A\mathbf{x} = \mathbf{b}$, where:
$A = \begin{pmatrix} 1 & -5 \\ 7 & -1 \end{pmatrix}$
and $\mathbf{b} = [-4 \;\; 6]^T$.
Start with the initial condition $\mathbf{x} = [0 \;\; 0]^T$ and use four iterations. Make sure that the method converges to the true solution. Discuss convergence of the Gauss-Seidel method for this case.

   (b) Solve the system with partial pivoting.

3. (a) Let: $A = \begin{pmatrix} 1 & 1 & 2 \\ 4 & 1 & 2 \\ -3 & 0 & 0 \end{pmatrix}$. Find an orthonormal basis for the vector space spanned by all vectors $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ is consistent. When the system is consistent, how many solutions does it have and why?

   (b) Let A be an $m \times n$ matrix. Prove that A and $A^T A$ have the same nullspace.

4. (a) Let $x_0, \ldots, x_n$ be distinct nodes and let f be a given real valued function with $n+1$ continuous derivatives. Let $P_n(x)$ be the Lagrange polynomial of degree n satisfying $P_n(x_i) = f(x_i)$ for $i = 0, \ldots, n$. Prove that for each x in the domain of f there is a $\xi \in \mathbb{R}$ with

$$f(x) - P_n(x) = \frac{(x - x_0) \ldots (x - x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

   (b) Prove that the function $g(x) = 1/(2 + x)$ has a unique fixed point x_* in $[0, 1]$ and describe the convergence of the iteration $x_{n+1} = g(x_n), x_0 = 1$.

5. Consider initial value problems of the form $y' = f(t, y), y(0) = y_0 \in \mathbb{R}$, where f is infinitely differentiable. Consider the family of explicit methods with uniform step size h given by

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + h\left[b_0 f(t_n, y_n) + b_1 f(t_{n-1}, y_{n-1})\right],$$

and suppose that $a_1 = 1 - a_0$.

(a) For which values of a_0, b_0 and b_1 is the method consistent?

(b) For which values of a_0, b_0 and b_1 will the method converge?

6. Suppose f is infinitely differentiable, and consider the definite integral

$$I(f) = \int_a^b f(x)dx.$$

Let $T_h(f)$ and $M_h(f)$ denote the composite trapezoidal and composite midpoint rule approximations of $I(f)$ with uniform subintervals of length h.

(a) Show that $T_{\frac{h}{2}}(f) = \frac{1}{2}\left(T_h(f) + M_h(f)\right)$.

(b) As $h \to 0$, the Euler-MacLaurin formula states that there is an asymptotic relation of the form

$$T_h(f) = I(f) + k_2 h^2 + k_4 h^4 + O(h^6),$$

where the constants k_2, k_4 are independent of h. Find a similar rule for $M_h(f)$.

(c) Apply one step of Richard extrapolation to derive a fourth order accurate quadrature formula using $M_h(f)$ and $M_{2h}(f)$.