Doctoral Qualifying Exam

Department of Mathematical Sciences

New Jersey Institute of Technology

Applied Math Part B: Real and Complex Analysis — May 2017

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Let $p > 0$, $f \in C([0,\infty))$, $u_0 \in \mathbb{R}$, and for $t \geq 0$ let

$$u(t) := u_0 e^{-pt} + \int_0^t e^{-p(t-\tau)} f(\tau) d\tau. \tag{1}$$

   (a) Show that $u \in C^1([0,\infty))$, and $u(t)$ solves the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + pu(t) = f(t), & t > 0, \\ u(t) = u_0, & t = 0. \end{cases} \tag{2}$$

   Carefully justify each step of your argument.

   Hint: Lebesgue dominated convergence theorem may be of help.

   (b) Show that every solution of Eq. (2) belonging to the class $C^1([0,\infty))$ is given by Eq. (1).

   (c) Show that $f \in C([0,\infty))$ does not imply that $u \in L^\infty(0,\infty)$. Is $u \in L^\infty(0,\infty)$ if $f \in L^\infty(0,\infty)$? If $f \in L^1(0,\infty)$?

2. Let $p > 0$, $r \in \mathbb{R}$, $g \in C([0,\infty)) \cap L^\infty(0,\infty)$, and let $u \in C^1([0,\infty))$ be a solution of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + pu(t) = rg(t)u(t), & t > 0, \\ u(t) = u_0, & t = 0. \end{cases} \tag{3}$$

   (a) Show that every solution $u \in C^1([0,\infty))$ of Eq. (3) satisfies

$$u(t) := u_0 e^{-pt} + r\int_0^t e^{-p(t-\tau)} g(\tau)u(\tau) d\tau. \tag{4}$$

   (b) Set up a fixed point iteration scheme for Eq. (4) and show that it defines a contraction map in the space $C([0,\infty))$ equipped with the usual sup-norm whenever $|r|\, \|g\|_{L^\infty(0,\infty)} < p$.

   (c) Is there a unique bounded solution to Eq. (3) for all $|r|$ sufficiently small depending on g and p?

   (d) Show that if $g \in C^1([0,\infty))$, then u solving Eq. (3) also belongs to $C^2(0,\infty)$.

3. Let $p > 0$, $f \in L^1(0,\infty)$, $u_0 \in \mathbb{R}$, and define

$$v(t) := \begin{cases} u(t) & t \geq 0, \\ 0 & t < 0, \end{cases} \tag{5}$$

   where $u(t)$ is given by Eq. (1).

(a) Show that $v \in L^1(\mathbb{R})$ and express its Fourier transform:

$$\widehat{v}(k) = \int_{\mathbb{R}} e^{-2\pi i k t} v(t)\, dt, \tag{6}$$

in terms of the Fourier transform of f extended by zero to the whole real line.

(b) In the rest of the questions, set $f = 0$ and assume $u_0 \neq 0$. Show that the inversion formula of Fourier transform:

$$v(t) = \int_{\mathbb{R}} e^{2\pi i k t}\, \widehat{v}(k)\, dk, \tag{7}$$

cannot be applied to $\widehat{v}$, if the integral is understood in the sense of Lebesgue.

(c) Show that the inversion formula may be applied to $\widehat{v}_R(k) := \widehat{v}(k)\chi_{(-R,R)}(k)$, where $\chi_{(-R,R)}$ is the characteristic function of $(-R, R)$, for any $R > 0$.

(d) Denoting the inverse Fourier transform of $\widehat{v}_R$ by v_R, show that $v_R \to v$ in $L^2(\mathbb{R})$.

4. Prove, using only the basic definition of differentiability, that the function defined as

$$F(z) = \int_C \frac{f(\zeta) d\zeta}{\zeta - z},$$

where f is continuous on the unit circle C, is analytic for $|z| < 1$, and find the formula for the derivative.

5. Consider the infinite series

$$F(z) := \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

(a) Show that F is uniformly convergent on $\Omega(\epsilon) := \{z \in \mathbb{C} : |z - n| \geq \epsilon, 0 < \epsilon < 1/4,$ for all $n \in \mathbb{Z}\}$, and therefore conclude that F is meromorphic on the whole complex plane, with all poles of order two.

(b) Explain why

$$\int_{|z|=R} F(z) dz = 0$$

for all $R > 0$ that are not positive integers.

6. Let $\Phi = \Phi(z, \mu) := 5z - \mu^2 \left(4z^4 + 1\right)$, where μ is a complex parameter. The following concern fixed points of Φ; that is, $z = z(\mu)$ such that $z = \Phi(z, \mu)$.

(a) Use Rouché's theorem to prove that Φ has a unique fixed point as long as $|z| \leq 1$ and $|\mu| < \sqrt{2}$ and $z(\mu)$ is an analytic function of the parameter.

(b) Use a variant of the zero-pole theorem for meromorphic functions ($\int_C (f'/f) dz = 2\pi i$(#zeros of f −#poles of f)) to find the power series representation of the solution $z = z(\mu)$ in (a).