Department of Mathematical Sciences
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Part A: Applied Mathematics — Doctoral Qualifying Exam, May 2017

The first three questions are based on Math 613 and the next three questions are about Math 651.

1. i) Write down the heat equations in a one-dimensional rod, $0 \le x \le 1$, composed of two materials, of equal size, with a different thermal conductivity coefficient, in perfect thermal contact, i.e. the temperature is continuous and no heat loss at where two materials join. Assume that in each material, physical properties are constant and there is no heat source. A constant temperature of T_0 is imposed at $x = 0$ and a constant temperature of T_1 is imposed at $x = 1$ with an initial temperature distribution $f(x)$.

   ii) Choose a characteristic length, time and temperature and nondimensionalize the equations.

   iii) Consider the thermal conductivity of material 1 to be 1 and the thermal conductivity of material 2 to be 2. Determine the equilibrium temperature distribution with $T_0 = 1$ and $T_1 = 0$.

2. Consider the flow of traffic described by

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

   where ρ is the traffic density and q the flux of traffic. Suppose that $q = \rho^2/2$ and the initial traffic density is

$$\rho(x,0) = \begin{cases} a & x < 1 \\ b & x > 1\,. \end{cases}$$

   Consider two cases $a > b$ and $a < b$, and determine the traffic density for $t > 0$ in each case. Explain your answers and sketch the characteristics in the x-t plane for each case.

3. Consider a linearly damped pendulum model

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \sin\theta = 0,$$

   with $\theta = \epsilon\alpha_0$ and $d\theta/dt = 0$ at $t = 0$, where $\epsilon \ll 1$. Using the perturbation method, find a solution correct to $O(\epsilon)$. Explain the behavior of your solution depending on the values of γ.

4. Find the general solution of the following ODEs.

   (a) $y'' + \frac{1}{x}y' - \frac{1}{x^2}y - x^2 = 0, \;\; x > 0.$

   (b) $yy'' + (y')^2 = 0$. (Hint: Substitute $y(x) = e^{u(x)}$.)

5. Use any method to find the solution of the wave equation

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < 1,\, t > 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x - x^2 \\ u_t(x,0) = 0 \end{cases}$$

6. Find the solution of the PDE

$$u_t = u_{xx} - \gamma u + g(x), \quad -\infty < x < \infty,\, t > 0$$

subject to the initial condition

$$u(x,0) = 0$$

where $\gamma > 0$ and

$$g(x) = xe^{-x^2/2}.$$

You may use the attached table of Fourier transforms. For full credit, your answer should consist of a single integral with a real-valued integrand.

Table of Fourier Transforms

$f(x)$	$F(k)=\mathcal{F}\{f(x)\}$	$f(x)$	$F(k)=\mathcal{F}\{f(x)\}$
$f(x)$	$\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk$	$F(k)$
$\delta(x)$	1	1	$2\pi\delta(k)$
$e^{-ax^2/2}$, $a>0$	$\sqrt{\frac{2\pi}{a}}e^{-k^2/(2a)}$	$e^{-a\|x\|}$, $a>0$	$\frac{2a}{a^2+k^2}$
$\frac{1}{a^2+x^2}$, $a>0$	$\frac{\pi}{a}e^{-a\|k\|}$	$\frac{x}{\|x\|}$	$\frac{2}{ik}$
$H(x)$	$\pi\delta(k)+\frac{1}{ik}$	$H(a-\|x\|)$	$\frac{2}{k}\sin(ak)$
$\frac{df}{dx}$	$ikF(k)$	$xf(x)$	$i\frac{dF}{dk}$
$f(x-a)$	$e^{-iak}F(k)$	$e^{iax}f(x)$	$F(k-a)$
$f(ax)$, $a\neq 0$	$\frac{1}{\|a\|}F\left(\frac{k}{a}\right)$	$f(x)*g(x)$	$F(k)G(k)$