Doctoral Qualifying Exam
Department of Mathematical Sciences
New Jersey Institute of Technology

Applied Math Part C: Linear Algebra and Numerical Methods May 2017

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. (a) Prove that for any A and $\mathbf{b}$, one and only one of the following systems has a solution: (i) $A\mathbf{x} = \mathbf{b}$, (ii) $A^T\mathbf{y} = \mathbf{0}$, $\mathbf{y}^T\mathbf{b} \neq 0$.

   (b) Prove that the nullspace of $n \times n$ matrix A and the nullspace of A^*A are the same. How are the ranks of the two matrices related?

2. (a) Show that $||A||_2 = \sqrt{\lambda_{max}(A^*A)}$.

   (b) Let A be a unitary matrix. Show that $cond_2(A) = 1$.

   (c) Prove that a unitary matrix is orthogonally diagonalizable.

3. (a) Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 4 \end{pmatrix}.$$
Using the inclusion principle, find bounds for the eigenvalues of A.

   (b) Let A be an $n \times n$ matrix that is diagonalizable. Let A have $\lambda = 0$ with algebraic multiplicity m. Find the rank of A, justifying all your statements.

4. Consider the function
$$g(x) = x + \frac{C(x^2 - x)}{Cx + 1}$$
where $C \in \mathbb{R}$ is a constant.

   (a) Find all fixed points of g.

   (b) For each fixed point α, find the range of values of C for which the iteration
$$x_{n+1} = g(x_n)$$
is guaranteed to converge given sufficiently small $|x_0 - \alpha| > 0$.

   (c) For each fixed point α, find a value of C that ensures that the iteration converges super-linearly.

5. Consider the approximation of the integral
$$I \equiv \int_0^h \sqrt{x} f(x)\, dx$$
for small $h > 0$.

(a) Let $I(h)$ be the two-point trapezoid rule approximation of this integral when $f(x) = 1$. Determine the *relative error* in the approximation.

(b) The usual error analysis for the two-point trapezoid rule suggests that

$$I = I(h) + \mathcal{O}(h^3).$$

Is this consistent with your answer from part (a)? Explain.

(c) For a general positive function $f \in C^2$, $f > 0$, find a two-point quadrature formula that approximates I with a *relative error* of $\mathcal{O}(h^2)$.

6. Consider the numerical solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0$$

using the two-step method

$$y_{n+2} = \frac{1}{3}y_n + \frac{2}{3}y_{n+1} + \frac{4}{3}hf(t_{n+1}, y_{n+1})$$

with the exact starting values y_0, y_1.

(a) Determine the leading term in the local truncation error. What is the order of the method?

(b) Is this a convergent method? Explain.

(c) Is this method A-stable? Explain.