Doctoral Qualifying Exam
Department of Mathematical Sciences
New Jersey Institute of Technology

Applied Math Part B: Real and Complex Analysis May 2015

The first three questions are about Real Analysis and the next three questions are about Complex Analysis.

1. Show that $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \tan^{-1} x$$

has no fixed point, and

$$|T(x) - T(y)| < |x - y| \quad \text{for all} x \neq y \in \mathbb{R}.$$

Why doesn't this example contradict the contraction mapping theorem?

2. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f. What does the Fourier series converge to? (Use complex form of the Fourier series $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$.)

(b) Show that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

(c) Deduce that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \to 0$ and show that

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

3. Consider

$$I = \int_0^{\infty} \frac{\log x}{x^{1/2}(1 + x^2)} f(x)\, dx.$$

where $f(x)$ is a discontinuous function that equals 1 a.e. on $[0, \infty)$.

Use Levi's monotone convergence theorem to prove that I exists as a Lebesgue integral.

4. **Series and singularities**

Let $\log_p(z)$ denote the principal branch of logarithm with $-\pi \leq \arg(z) < \pi$. Find and categorize the isolated singularity of the following function, and determine its residue at this isolated singularity. Describe the domain of convergence of the Laurent series centered at this singularity:

$$f(z) = \frac{1}{\log_p(\log_p(z))}$$

5. **Complex integration and residues**

Let $\phi \in (0, \pi)$ and p denote real constants. Prove the integration result

$$\int_0^\infty \frac{r^p \; dr}{r^2 + 2r\cos(\phi) + 1} = \frac{\pi}{\sin(p\pi)} \frac{\sin(p\phi)}{\sin(\phi)}$$

by integrating $f(z) = z^p / \left(z^2 - 2z\cos(\phi) + 1\right)$ over a "keyhole" contour traversing the branch cut of z^p on the **negative** real axis (choose $-\pi \leq \arg(z) < \pi$). Finally, determine conditions on the value of p required for the convergence of this integral. Note: the sign of the second term in the denominator of $f(z)$ changes from negative to positive when integrating along the negative real axis.

6. **The Argument Principle and the Rouché's Theorem**

(a) Use Rouché's Theorem to find the number of roots of function $f(z) = e^z + 6z^3 - z^5 - iz$ located within the unit disk, $|z| < 1$. Make sure to explain clearly.

(b) Use the Argument Principle and your answer to part (a) of this problem to determine the value of the following integral over the unit circle:

$$\oint_{|z|=1} \frac{e^z + 18z^2 - 5z^4 - i}{e^z + 6z^3 - z^5 - iz} dz$$

Hint: do **not** examine the winding number, it's much easier to use the other part of the Argument Principle!