Doctoral Qualifying Exam
Department of Mathematical Sciences
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Applied Math Part C: Linear Algebra and Numerical Methods May 2015

The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. (a) Let A be a Hermitian matrix. Prove that its eigenvalues are real.
   (b) Let A be a Hermitian matrix. Prove that eigenvectors corresponding to distinct eigenvalues are orthogonal.
   (c) Let U be a unitary matrix. Prove that the modulus of its eigenvalues is 1.
   (d) If $\mathbf{u}$ is a unit vector, show that matrix $A = I - 2\mathbf{u}\mathbf{u}^*$ is a normal, unitary matrix. Symbol $*$ denotes conjugate transpose.
   (e) Let A, B, and C be $n \times n$ non-singular matrices such that $AB = I$ and $CA = I$. Show that $B = C$.

2. (a) Find a basis for the space in which $\mathbf{b} = [b_1\ b_2\ b_3]^T$ needs to lie for the following system to be solvable:

$$x + 2y = b_1$$

$$x + y = b_2$$

$$2x + 2y = b_3$$

   (b) Prove that for any A and $\mathbf{b}$, one and only one of the following systems has a solution: (i) $A\mathbf{x} = \mathbf{b}$, (ii) $A^T\mathbf{y} = \mathbf{0}$, $\mathbf{y}^T\mathbf{b} \neq 0$.

3. (a) Let: $A = \begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$. Determine whether the matrix is diagonalizable. If it is, express A in the form $A = P\Lambda P^{-1}$, where P has the eigenvectors of A as columns and Λ is a diagonal matrix with the eigenvalues of A on the diagonal.
   (b) Let: $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. Determine whether the matrix is diagonalizable and whether it can be expressed in the form $A = U\Lambda U^*$, where U has the orthonormal eigenvectors of A as columns and Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

4. Consider finding the root $x = 0$ of $f(x) = \sin x - x$.
   (a) Apply Newton's method to this problem and write down explicit iteration formula in the form $x_{n+1} = F(x_n)$. Based on the the type of root at $x = 0$, discuss what order of convergence is expected.
   (b) Formulate the problem as a fixed point iteration $x_{n+1} = g(x_n)$. Find $g'(x)|_{x=0}$ and discuss whether the value you find is consistent with the type of root found in (a).

(c) Formulate modified Newton method characterized by quadratic order of convergence by using the information about the type of root at $x = 0$. Write down modified iteration scheme.

5. Function $f(x)$ is specified at the points $x_0,\ x_0 + h,\ x_0 + 2h$.

   (a) Using the above information, find as accurate approximation as possible to the second derivative of $f(x)$ at $x = x_0$.

   (b) Find explicit expression for the truncation error, E, of the scheme. That is, if $E = Ch^n$, find C and n.

6. Let $\theta \in [0, 1]$ be a constant and denote $x_{n+\theta} = (1 - \theta)x_n + \theta x_{n+1}$. Consider the generalized midpoint method

$$y_{n+1} = y_n + hf(x_{n+1}, (1 - \theta)y_n + \theta y_{n+1})$$

and its trapezoidal analogue

$$y_{n+1} = y_n + h[(1 - \theta)f(x_n, y_n) + \theta f(x_{n+1}, y_{n+1})]$$

Apply these schemes to the model problem

$$\frac{dy}{dx} = \lambda y$$

with λ a negative constant and find the range of absolute stability for θ's in the specified range. Are there the values of θ for which the methods are A-stable?