Linear algebra qual — May 2024

1. Similar matrices.

   (a) Prove that if M_1 and M_2 are similar then $\dim\ker(M_1) = \dim\ker(M_2)$.

   (b) Is it true that if $\dim\ker(M_1) = \dim\ker(M_2)$ then M_1 and M_2 are similar?

   (c) Divide the matrices below into sets so that all matrices within each set are similar. Make the sets as large as possible and explain your reasoning. You can use any facts you know about similar matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$E = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, F = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}, G = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

   (a) Prove that the set of matrices

$$W = \{B \in \mathbb{R}^{2\times 2} \text{ such that } AB = BA\}$$

   is a subspace of $\mathbb{R}^{2\times 2}$.

   (b) Prove that W has dimension 2.

   (c) Show that the matrix A^{-1} can be written as $A^{-1} = \alpha I + \beta A$ for some $\alpha, \beta \in \mathbb{R}$.

3. Consider the data

t	-2	-1	0	1	2
y	15	17/2	3	3/2	7

   (a) Find a model of the form $y(t) = \alpha + \beta t$. Determine the coefficients α and β using a least squares fit.

   (b) Find a model of the form $y(t) = \alpha + \beta t + \gamma t^2$. Determine the coefficients α, β and γ using a least squares fit.