DOCTORAL QUALIFYING EXAM, JAN 2019

Department of Mathematical Sciences

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Part A: Applied Statistics - Distribution Theory & Regression Analysis.

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1. Suppose that (X_1, X_2) is distributed bivariate normal with means, variances and correlation $E(X_1) = E(X_2) = 0, Var(X_1) = Var(X_2) = 1, \rho = 1/3$, respectively. Let $Y = X_1 X_2$. Compute the moment generating function (mgf) of Y and show that the mgf exists in the neighborhood of zero.

2. Suppose that X is distributed as Poisson λ. Compute the k^{th} order factorial moment $E[X(X-1)(X-2)(X-3)(X-[k-1])]$, where k is any positive integer greater than zero. Give justification for the method used.

3. Let X and Y be any two random variables, derive $E\,|XY| \leq \sqrt{E(X^2)E(Y^2)}$.

4. Consider the usual multiple linear regression model, written in matrix notation as $Y = X\beta + \epsilon$ for $\epsilon \sim N(0, \sigma^2 I)$. Assume that $X = (\mathbf{1}, X^{\star})$ (where $\mathbf{1} = (1, 1, \ldots, 1)'$) has full rank. Recall that the various sums of squares from the ANOVA table for this model: SSTotal, SSReg and SSE. As is well-known, these sums of squares are quadratic forms.

   (a) Show that in each case, the matrix of the quadratic from is symmetric and idempotent.

   (b) To test the hypothesis $H_0 : \beta_2 = 0$ vs $H_1 : \beta_2 \neq 0$ for a normal error linear model with two predictors X_1 and X_2: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$ for $\epsilon \sim N(0, \sigma^2 I)$, we fit a full model with and a reduced model with only X_1 in it and a full model with both X_1 and X_2 in it. Prove the independence of the extra sum of squares $SSR(X_1)$ and $SSR(X_2|X_1)$.

(c) Derive the distribution of the partial F test statistics for the hypothesis $H_0 : \beta_2 = 0$ vs $H_1 : \beta_2 \neq 0$ using the extra sum of squares for data satisfying the normal error linear model with two predictors X_1 and X_2: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$ for $\epsilon \sim N(0, \sigma^2 I)$.

5. Let $\beta_1, \beta_2, \beta_3$ be the interior angles of a triangle, so that $\beta_1 + \beta_2 + \beta_3 = 180$ degrees. Suppose we have available estimates Y_1, Y_2, Y_3 of $\beta_1, \beta_2, \beta_3$, respectively. We assume that $Y_i \sim N(\beta_i; \sigma^2), i = 1, 2, 3$ (σ^2 is unknown) and that the Y_i's are independent. What is the F-test for testing the null hypothesis that the triangle is equilateral?

6. Let $Y \sim N(\mu, \Sigma)$, $r = Y'AY$, and $s = BY$, where A is a symmetric and idempotent matrix.

   (a) Prove that if $B\Sigma A = 0$ then r and s are independent.

   (b) Use the result from part a to show that the sample mean and sample variance of Y are independent.