DOCTORAL QUALIFYING EXAM, JAN 2019
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Part A: Applied Mathematics - Problems 1-3 are based on Math 613 and 4-6 based on Math 651.

Problem 1

Consider the following PDE describing the diffusion of Ca^{2+} concentration field $C(\mathbf{r}, t)$ and its reaction with buffer concentration field $B(\mathbf{r}, t)$:

$$\frac{\partial C}{\partial t} = D\ \nabla^2 C - k^+ C \cdot B + k^- (B_T - B)$$

(a) What are the physical units of constants D, B_T, k^+ and k^-? Assume that the concentration fields C and B are given in units of reciprocal (inverse) volume.

(b) Non-dimensionalize this system, reducing the number of parameters as much as possible. Use the same length scale for all spatial variables ($[x] = [y] = [z] = L$), and the same scale for both concentration fields ($[C] = [B]$).

(c) Show two more independent choices for the length scale L that can be formed using constants D, B_T, k^+ and k^-, apart from your choice of L that you used in part (b).

Problem 2

Consider the following chemical reaction system:

$$A + A \overset{k_D}{\to} \varnothing$$

$$\varnothing \overset{k_B}{\to} A$$

(a) Write down the Chemical Master Equations for this reaction system.

(b) Write down the ODE for the evolution of the first moment of the number of A molecules (hint: multiply the Master Equations by n, and sum over all n).

(c) Write down the PDE for the probability-generating function, $F(z, t) = \sum_{n=0}^{\infty} p_n(t) z^n$ (do not solve!)

Problem 3

Consider a one-dimensional traffic flow model (assume all variables have been non-dimensionalized):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}[\rho u(\rho)] = 0 \quad (t > 0, -\infty < x < \infty)$$

Assume quadratic velocity-density dependence, $u(\rho) = 1 - \rho^2$, with initial traffic density given by

$$\rho(x_0, 0) = \begin{cases} 1, & \text{if } x_0 \leq 0 \\ 1/\sqrt{1 + x_0}, & \text{if } x_0 > 0 \end{cases}$$

(a) Find and plot characterisitc curves corresponding to $x_0 = -1$, $x_0 = 0$, $x_0 = 1$ and $x_0 = 2$. Does wave break-up occur in this case? Is there a location x at which solution $\rho(x, t)$ is constant?

(b) **Without solving** this PDE explicitly, make a rough plot of the traffic density at $t = 0$ and at $t = 1$.

(c) Find an explicit solution of this traffic/advection equation, $\rho(x, 1)$, at $t = 1$.

Problem 4

Consider the ordinary differential equation given by

$$(\sin x)y'' + 2(\cos x)y' - (\sin x)y = 0.$$

(a) Show that $x = 0$ is a regular singular point.

(b) Show that both linearly independent solutions can be written in the form of power series about $x = 0$.

(c) Find the first two nonzero terms in each power series.

Problem 5

Use the method of characteristics to solve

$$\frac{\partial u}{\partial x} - y^3 \frac{\partial u}{\partial y} = 1$$

where the Cauchy data are specified by $u = 1 + y^2$ along the line $x = 0$ for $-\infty < y < \infty$. Sketch the domain on which the solution is defined.

Problem 6

Consider the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - bx^2 \qquad \text{for} \qquad 0 < x < 1, \quad t > 0.$$

Assume boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = 0 \qquad \text{and} \qquad u(1, t) = 1$$

and initial condition $u(x, 0) = f(x)$.

(a) Use any means you choose to find the solution $u(x, t)$.

(b) Suppose we let

$$a_n(t) = \int_0^1 u(x, t) \cos\left((n - \frac{1}{2})\pi x\right) dx.$$

For any given $n > 0$, what value of $b = b_n$ is such that $a_n \to 0$ as $t \to \infty$?