Department of Mathematical Sciences
New Jersey Institute of Technology

Part A: Applied Mathematics — Doctoral Qualifying Exam, January 2017

The first three questions are based on Math 613 and the next three questions are about Math 651.

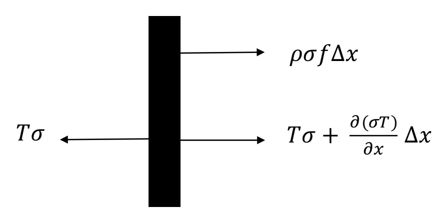
1. Consider the initial value problem

$$\frac{\partial \rho}{\partial t} - \rho \frac{\partial \rho}{\partial x} = -\alpha \rho$$
$$\rho(x,0) = f(x)$$

where α is a positive constant, and $-\infty < x < \infty$ and $t \geq 0$.
(a) Explain, in the context of traffic flow problem, what does this initial value problem represent.
(b) Find the general solution.
(c) Show that no shocks form for all time if $f'(x) \leq \alpha$.
(d) Find the explicit solution for $\alpha = 2$ and $f(x) = x$.

2. Consider a scalar quantity $M(\mathbf{x}, t)$ with density $u(\mathbf{x}, t)$, flux $\mathbf{q}(\mathbf{x}, t)$, and source density $Q(\mathbf{x}, t)$.
(a) Write the integral form of the conservation of $M(\mathbf{x}, t)$ expressed for any fixed spatial region Ω with the boundary $\partial\Omega$.
(b) Take u to be the mass density, $\mathbf{q} = -\mu \nabla u$ the mass flux, and $Q = ku(a-u)$ the mass rate per unit volume at which the mass is generated, where μ, k, a are positive constants. Write the resulting PDE.
(c) An example of this PDE is the reaction-diffusion equation that arises as a model in population dynamics, for example. Now consider the one-dimensional reaction-diffusion equation. Nondimesionalize the equation on a finite domain of length L by taking the appropriate scales.
(d) Give a physical interpretation of the nondimensional parameter that appears as the result of the nondimensionalization. Discuss small and large limits of the dimensionless number.
(e) There exist traveling wave solutions of the form $u(x,t) = f(x - ct)$, where $c > 0$ is a constant wave speed. Find the ODE that f satisfies by plugging f into the PDE. Find the equilibria of the ODE.

3. Consider the one-dimensional deformation of a thin slice of a loaded bar with the forces acting on it as shown below



where f is a body force per unit mass, T is the stress, and σ is the area of the cross section.
(a) Apply the Newton's second law of motion and find the linearized momentum balance equation in terms

of displacement.
(b) The strain is defined as

$$\frac{\text{present length - original length}}{\text{original length}}$$

Find an expression for the (linear) strain in spatial coordinates for when the displacement and its gradient are small.
(c) Use Hooke's law stress-strain relation in your derivation above and show that in the absence of the body force and when material properties and cross section area are constant, the differential equation reduces to the wave equation for the displacement.

4. Consider the ODE

$$2x^2y'' + (x - x^2)y' - y = 0.$$

Find the first four non-zero terms in each of two linearly independent series solutions about the point $x_0 = 0$.

5. (a) Consider the wave equation

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 2,\, t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = \phi(x), & 0 < x < 2 \\ u_t(x,0) = 0, & 0 < x < 2 \end{cases}$$

where

$$\phi(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

Sketch the solution $u(x,t)$ at $t = 1/2$. You do *not* need to solve the PDE.

(b) Use any method to find the solution $u(x,t)$ of the wave equation

$$\begin{cases} u_{tt} = u_{xx}, & x \in \mathbb{R},\, t > 0 \\ u(x,0) = 0, & x \in \mathbb{R} \\ u_t(x,0) = xe^{-x^2/2}, & x \in \mathbb{R}. \end{cases}$$

6. Consider the following boundary value problem in the square:

$$\begin{cases} u_{xx} + u_{yy} + \epsilon^2 u = 0, & 0 < x < 1,\, 0 < y < 1 \\ u_y(x,0) = u_y(x,1) = u_x(0,y) = 0 \\ u_x(1,y) = f(y) \end{cases}$$

where $0 < \epsilon < 1$ and f is continuous.

(a) Use the method of separation of variables to find the solution of the boundary value problem.
(b) If $\epsilon = 0$, what additional conditions should $f(y)$ satisfy to ensure existence of a solution?