Department of Mathematical Sciences
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Part A: Applied Mathematics — Doctoral Qualifying Exam, January 2016

The first three questions are based on Math 613 and the next three questions are about Math 651.

1. The incompressible Navier-Stokes equations are a model for fluid flow. Consider a channel geometry $0 \le y \le H$, $0 \le x \le \beta H$, where $H > 0$ is the height of the channel and $\beta > 0$ is the aspect ratio (i.e. $\beta =$ length / height). The fluid is modeled by pressure and velocity variables (u, v, p):

$$\begin{aligned} \rho(u_t + uu_x + vu_y) &= -p_x + \mu(u_{xx} + u_{yy}), \\ \rho(v_t + uv_x + vv_y) &= -p_y + \mu(v_{xx} + v_{yy}), \\ u_x + v_y &= 0. \end{aligned} \tag{1}$$

Here $[\rho] = ML^{-3}$, $[\mu] = L^{-1}T^{-1}M$ are the fluid density and viscosity respectively and M, L, T stand for units of mass, length and time. The boundary conditions associated with (1) in a channel geometry are no-flux and no-slip at the top and bottom of the channel:

$$u(x, 0, t) = u(x, H, t) = v(x, 0, t) = v(x, H, t) = 0, \quad \text{for all } 0 \le x \le \beta H,\ t > 0. \tag{2}$$

(a) Suppose a pressure $p(0, y, t) = p_0 > 0$, $p(\beta H, y, t) = 0$ is applied across the channel. Pressure has units $[p_0] = ML^{-1}T^{-2}$. Non-dimensionalize the equations (1) as follows.

- Use the height of the strip H as a length scale.
- Use the parameters ρ, p_0, H to introduce a time scale. Use this time scale to non-dimensionalize the derivatives ∂_t and along with the length scale H non-dimensionalize the variables u, v.
- Use p_0 to non-dimensionalize the pressure p.
- For your non-dimensional variables use X, Y, τ for space and time and U, V, P for the dependent variables.

Your new equations should have the form:

$$\begin{aligned} U_\tau + UU_X + VU_Y &= -P_X + \mathrm{Re}^{-1}(U_{XX} + U_{YY}), \\ V_\tau + UV_X + VV_Y &= -P_Y + \mathrm{Re}^{-1}(V_{XX} + V_{YY}), \\ U_X + V_Y &= 0. \end{aligned} \tag{3}$$

Explicitly state the dimensionless Reynolds number $\mathrm{Re} > 0$ in terms of the problem parameters ρ, μ, H, p_0.

(b) State the boundary conditions for U and V that correspond to the dimensional boundary conditions (2). State also the boundary conditions for the pressure P.

(c) Look for a steady state critical point solution. Specifically seek an ansatz, $V(X, Y, \tau) = V(Y)$ and $U(X, Y, \tau) = U(Y)$ depend on Y **ONLY** and $P(X, Y, \tau) = (1 - \beta^{-1}X)$. Find the simplified equations that $U(Y), V(Y)$ satisfy. These equations should involve only derivatives in Y.

(d) Solve for $U(Y)$, $V(Y)$. Hint: using the divergence equation $U_X + V_Y = 0$ and boundary conditions to solve for $V(Y)$ first.

(e) Find the flux of the steady solution you found in part (d) (i.e. the total fluid per time that crosses a slice $0 \le y \le H$ at any x location). You may compute and express the flux in terms of the non-dimensional variables.

2. A collection of particles diffuse via the diffusion equation

$$u_t = D\, u_{xx}, \quad -\infty < x < \infty, t > 0. \tag{4}$$

For this question you may assume that all solutions are smooth and vanish, along with all of their derivatives, rapidly at infinity i.e. $\lim_{x\to\pm\infty} |x|^m\, u(x,t) = 0$, and $\lim_{x\to\pm\infty} |x|^m\, u_x(x,t) = 0$ for any $m \ge 1$.

(a) Given that $u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is the solution to the diffusion equation with $D = 1$ and initial data $u(x,0) = \delta(x)$. By rescaling time, find the solution $u(x,t)$ to (4) for arbitrary $D > 0$.

(b) Suppose instead that the particles start with an initial density profile $u(x,0) = f(x)$ for some function $f(x)$. Show by any means that the first moment is constant in time:

$$\int_{-\infty}^{\infty} s\, u(s,t)\, \mathrm{d}s = \int_{-\infty}^{\infty} s f(s)\, \mathrm{d}s \tag{5}$$

Conclude that if $f(s)$ is an even function, the first moment is 0 for all time. (Physically this says that particles are equally likely to diffuse to the right and to the left).

(c) Now consider the second moment. The second moment physically represents the "average" spread or width of the solution.

$$M_2(t) = \int_{-\infty}^{\infty} s^2 u(s,t)\, \mathrm{d}s, \tag{6}$$

show that $M_2(t) = 2Dt + M_2(0)$, where $M_2(0)$ is the second moment at time $t = 0$. Conclude that the average "position width" of a droplet of dye behaves as

$$\text{Width } = \sqrt{M_2(t)} = \sqrt{2Dt + M_2(0)}. \tag{7}$$

3. Consider the Burgers' equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = 0 \tag{8}$$

with initial data

$$U(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

(a) For each of the three regions $x_0 < 0$, $0 \le x_0 \le 1$ and $x_0 > 1$, find the equations for the characteristics. Write the initial value of x_0 in terms of points x, t.

(b) The characteristics collide at some time t_1. What is t_1?

(c) Find a closed form solution for $U(x,t)$ in every region, for all times $0 \le t < t_1$.

(d) For $t \ge t_1$ a shock develops. Let $x_s(t)$ be the position of the shock front. Find $x_s(t)$ for all time $t \ge t_1$.

(e) Suppose now that instead of $U(x,0) = 1 - x$ for $0 \le x \le 1$, the initial data is

$$U(x,0) = \begin{cases} 1 & x < 0 \\ (1-x)^\alpha & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

where $\alpha \ge 1$. Find the time t_1 that it takes for a shock to form in terms of only α. What happens when $\alpha \to \infty$?

4. Let

$$F(x) = \int_x^\infty e^{-t^2} dt.$$

Show that

$$F(x) \sim \frac{e^{-x^2}}{2x}, \quad x \to +\infty.$$

5. The Airy function $\mathrm{Ai}(x)$ is the unique solution of

$$\frac{d^2y}{dx^2} - xy = 0$$

which satisfies

(i) $\lim_{x\to\pm\infty} y = 0$

(ii) $y(0) = 3^{-2/3}/\Gamma(2/3) = 3^{-2/3}\Gamma(1/3)\sqrt{3}/(2\pi) = \frac{1}{\pi}\int_0^\infty \cos(\omega^3/3)d\omega$.

(a) Compute the Fourier transform of the Ai function.

(b) Use the previous part to determine an integral representation of the Ai function.

(c) Use the method of stationary phase to determine the leading asymptotic behavior of the Ai function as $x \to +\infty$.

(d) Determine the same asymptotic behavior starting from the differential equation.

6. Let $u(x,t)$ solve the wave equation

$$\begin{cases} u_{tt} - \Delta u & = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \varphi(x),\ u_t(x,0) & = 0 \quad \text{on} \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

where the function φ is smooth and compactly supported. Show that the function

$$v(x,t) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^\infty e^{-\frac{s^2}{4t}} u(x,s) ds$$

satisfies the initial value problem for the heat equation

$$v_t - \Delta v = 0,\ t > 0,\ v(x,0) = \varphi(x).$$