Math 111 Final Exam
December 15, 2017

Time: 2 hours and 30 minutes
Instructions: Show all work for full credit.
No outside materials or calculators allowed.
Extra Space: Use the backs of each sheet for extra space. Clearly label when doing so.

Name: ______________________________

ID #: ______________________________

Instructor/Section: ___________________

"I pledge by my honor that I have abided by the NJIT Academic Integrity Code."

__________________________ (Signature)

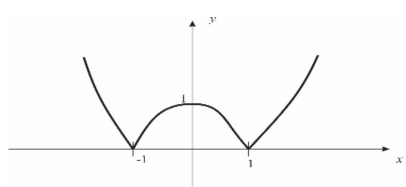
Problem	Value	Score
1	10 pts.	
2	10 pts.	
3	15 pts.	
4	10 pts.	
5	15 pts.	
6	10 pts.	
7	15 pts.	
8	15 pts.	
TOTAL	100	

1. Consider the function $y = f(x) = |x^2 - 1|$.

(a) Sketch *f*, noting that $x^2 - 1$ is negative for $-1 < x < 1$. (Hint: Plot $y = x^2 - 1$.) **(3 pts.)**
(b) Show that the function is continuous everywhere. **(2 pts.)**
(c) Show $f'(x)$ is not continuous at $x = 1$ by showing $\lim_{x\to 1-} f'(x) \neq \lim_{x\to 1+} f'(x)$. **(5 pts.)**

Solution 1.

(a) The sketch is



(b) As the function is equal to $x^2 - 1$ for $|x| > 1$ and $1 - x^2$ for $|x| < 1$, which are continuous, it is only necessary to prove continuity at $x = \pm 1$. Since $x^2 - 1 \to 0$ as $x \to -1^-, 1^+$ and $1 - x^2 \to 0$ as $x \to -1^+, 1^-$, we see $\lim_{x\to -1} f(x) = \lim_{x\to 1} f(x) = 0 = f(-1) = f(1)$, so the continuity follows.

(c) As $f'(x) = 2x$ for $x > 1$ and $f'(x) = -2x$ for $0 < x < 1$, we compute that

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} (-2x) = -2 \neq 2 = \lim_{x\to 1+} f'(x) = \lim_{x\to 1+} (2x).$$

2. Let $y = y(x)$ be implicitly defined by $y = 1 + e^x \sin(xy)$.

(a) Compute the derivative *dy*/*dx*. **(5 pts.)**

(b) Find the tangent line to the curve at $(\pi, 1)$. . **(5 pts.)**

Solution 2.

(a) Assuming $y = y(x)$ is differentiable, we compute by implicit differentiation that

$$\frac{dy}{dx} = e^x \sin(xy) + e^x \cos(xy)\left[y + x\frac{dy}{dx}\right] \Rightarrow \left[1 - xe^x \cos(xy)\right]\frac{dy}{dx} = e^x\left[\sin(xy) + y\cos(xy)\right] \Rightarrow$$

$$\frac{dy}{dx} = \frac{e^x\left[\sin(xy) + y\cos(xy)\right]}{1 - xe^x \cos(xy)}.$$

(b) At $(\pi, 1)$ it follows from (a) that

$$\frac{dy}{dx}(\pi) = \frac{e^\pi\left[\sin\pi + (1)\cos\pi\right]}{1 - \pi e^\pi \cos\pi} = \frac{-e^\pi}{1 + \pi e^\pi}.$$

Hence, the tangent line at $(\pi, 1)$ is

$$y - 1 = \frac{-e^\pi (x - \pi)}{1 + \pi e^\pi} = \frac{e^\pi (\pi - x)}{1 + \pi e^\pi}.$$

3. (a) A rectangular yard is to be completely enclosed by fencing and then divided into three enclosures of equal area by fences parallel to and of the same length as one side of the yard. If 400 ft. of fencing is available, what dimensions maximize the enclosed area? Verify that your answer is the largest possible area. **(8 pts)**

(b) Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible. Verify that your answer is the smallest possible sum. **(7 pts.)**

Solution 3.

(a) Let x be the length of each of the four parallel fences for the three equal enclosures and $6y$ be the lengths of the remaining fencing. Then, we are given that

$$4x + 6y = 400.$$

We have to maximize the area $A = x(3y)$ subject to the above constraint, so we maximize

$$A = A(x) = x(200 - 2x) = 2x(100 - x)$$

for $0 < x < \infty$. Noting that $A(0) = A(100) = 0$ and that A is negative for $x > 100$, it follows that we can confine our attention to the closed interval [0,100] on which A has a maximum since it is continuous on the whole real line. The maximum must occur at an interior point of the interval which must be a critical point, so we compute that

$$A' = 200 - 4x = 0 \Rightarrow x = 50 \Rightarrow y = 100/3,$$

so it follows that the maximum area of the rectangular yard is 50×100 ft.^{2}. It is a local maximum by the first derivative or second derivative test (the first derivative changes sign from positive to negative across $x = 50$ and the second derivative is -4, so the graph is concave down on the interval [0,100]). It is the only critical point of $A(x)$, so it must yield a global maximum.

(b) Here we need to find the minimizer of $S(x) = (1/x) + 4x^2$ for $x > 0$. Noting that $S(x) \to \infty$ as $x \to 0^+, \infty$, we conclude that S must have a minimum on a bounded closed positive interval. Moreover, the function is infinitely differentiable for $x > 0$, so the minimum must occur at a critical point. Now

$$S' = -x^{-2} + 8x = 0 \Rightarrow x = 2,$$

and $S''(x) = 2x^{-3} + 8$, which is concave up for all $x > 0$. Consequently, the absolute (global) minimum is

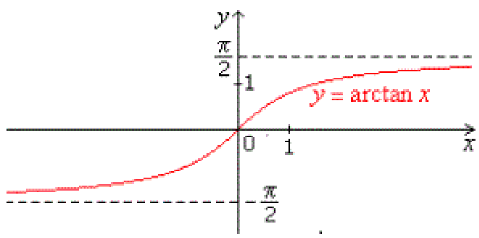
$$S(2) = (1/2) + 16 = 33/2,$$

which is attained at $x = 2$.

4. Consider the function $y = f(x) = \dfrac{\arctan x}{1+x^2} = \dfrac{\tan^{-1} x}{1+x^2}$.

(a) Use the derivative to show that the function has its maximum and minimum at the positive and negative solutions, respectively, of $\tan\left(1/(2x)\right) = x$, which happen to be at $x \approx \pm 0.765$. Show your work and state any tests used. **(7 pts.)**

(b) Sketch the graph of the function using the definition and the above information. (Hint: the graph of arctan*x* shown below should be helpful here and for (a)). **(3 pts.)**



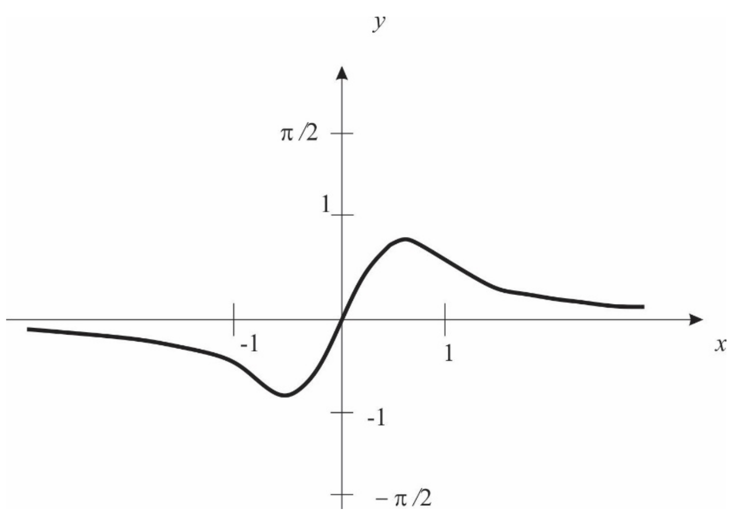
Solution 4.

(a) The function is clearly bounded and has continuous derivatives of all orders on the real line. Thus it has a maximum and minimum at critical points (where the derivative vanishes). Using the quotient rule and the usual formulas, we compute that

$$y' = f'(x) = \frac{(1+x^2)(1+x^2)^{-1} - 2x\tan^{-1}x}{(1+x^2)^2} = 0 \Rightarrow \tan^{-1}x = 1/2x \Leftrightarrow \tan(1/2x) = x.$$

This has symmetric solutions, with the maximum at $x_* > 0$ and the minimum at $-x_*$.

(b) The graph has the form shown below.



5. Consider the function $y = f(x) = \dfrac{x^2}{1+x^2}$.

(a) Find and classify (as local or global maxima or minima) any critical points. **(3 pts.)**
(b) Find all asymptotes of the curve. **(3 pts.)**
(c) Find all inflection points. **(3 pts.)**
(d) Find where the function is increasing and decreasing. **(3 pts.)**
(e) Sketch the curve using the above information. **(3 pts.)**

Solution 5.

(a) The function is differentiable on the whole real line, so the only critical points are zeros of the derivative:

$$y' = f'(x) = \frac{(1+x^2)(2x)-(2x)x^2}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2} = 0 \Rightarrow x = 0.$$

Thus *f* has a global minimum of 0 at $x = 0$ since the function is decreasing (increasing) for $x < 0$ ($x > 0$).

(b) There are no vertical or oblique asymptotes, but $y = 1$ is a horizontal asymptote since $f(x) \to 1$ as $x \to \pm\infty$.

(c) As *f* is an even function, it suffices to consider the second derivative for $x > 0$. We compute that

$$y'' = f''(x) = 2\frac{d}{dx}\left(\frac{x}{(1+x^2)^2}\right) = \frac{2}{(1+x^2)^4}\left\{(1+x^2)^2 - 2(1+x^2)(2x)x\right\} = \frac{2(1-2x^2-3x^4)}{(1+x^2)^4},$$

which starts out positive at $x = 0$ and changes to negative when

$$u^2 + (2/3)u - 1 = 0,$$

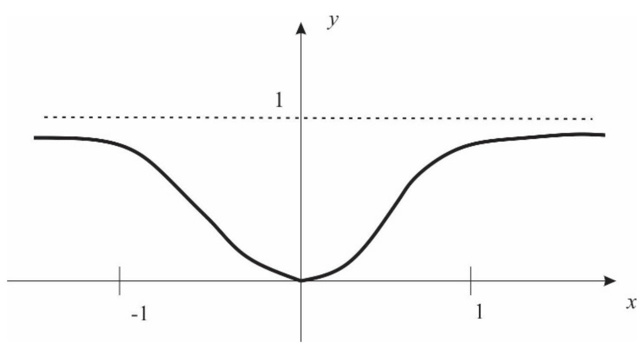
where $u = x^2$. The quadratic formula yields

$$u = x^2 = \frac{1}{2}\left\{-\frac{2}{3} \pm \sqrt{\frac{4}{9}+\frac{4}{3}}\right\} = \frac{1}{3}\left[-1 \pm \sqrt{5}\right] \Rightarrow x = \frac{1}{\sqrt{3}}\sqrt{\sqrt{5}-1}$$

is the positive point of inflection, with a matching point of inflection at $x = -(3)^{-1/2}\left(\sqrt{5}-1\right)^{1/2}$.

(d) As shown in (a), the function is decreasing $(f' < 0)$ for $x < 0$ and increasing $(f' > 0)$ for x < 0.

(e) The sketch of the graph is



6. Evaluate each of the following limits:

   (a) $\lim_{x\to 0} \dfrac{\tan x}{\arctan x} = \lim_{x\to 0} \dfrac{\tan x}{\tan^{-1} x}$. **(3 pts.)**

   (b) $\lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$. **(4 pts.)**

   (c) $\lim_{x\to 0+} x^x$. **(3 pts.)**

Solutions 6.

(a) This is a 0/0 indeterminate, so we can apply l'Hôpital's rule to obtain

$$\lim_{x\to 0} \frac{\tan x}{\arctan x} = \lim_{x\to 0} \frac{\tan x}{\tan^{-1} x} = \lim_{x\to 0} \frac{(\tan x)'}{(\tan^{-1} x)'} = \lim_{x\to 0} \frac{\sec^2 x}{(1+x^2)^{-1}} = \frac{1}{1} = 1.$$

(b) Let $u = (1+2x)^{1/(2\ln x)}$ and $v = \ln u = \dfrac{\ln(1+2x)}{2\ln x}$ and first compute $\lim_{x\to\infty} v$. This yields an ∞/∞ indeterminate, so l'Hôpital's rule is applicable:

$$\lim_{x\to\infty} v = \lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty} \frac{(\ln(1+2x))'}{(2\ln x)'} = \lim_{x\to\infty} \frac{2/(1+2x)}{2/x}$$

$$= \lim_{x\to\infty} \frac{x}{(1+2x)} = \lim_{x\to\infty} \frac{x}{x((1/x)+2)} = \lim_{x\to\infty} \frac{1}{1((1/x)+2)} = \frac{1}{2}.$$

Hence, $\ln u \to 1/2$ as $x \to \infty$, so $\lim_{x\to\infty} u = e^{1/2} = \sqrt{e}$.

(c) Let $u = x^x$ and $v = \ln u$. We first compute

$$\lim_{x\to 0+} v = \lim_{x\to 0+} (x\ln x) = \lim_{x\to 0+} \frac{\ln x}{1/x} \to \frac{-\infty}{\infty},$$

so we can use l'Hôpital's rule:

$$\lim_{x\to 0+} v = \lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{(\ln x)'}{(1/x)'} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = 0,$$

which means that $\lim_{x\to 0+} u = \lim_{x\to 0+} x^x = e^0 = 1$.

7. The function $f(x)$ is differentiable. In the table below are some values of $f(x)$ and its integrable derivative $f'(x)$.

x	1	2	3	4	5	6
$f(x)$	3	2	0	1	4	5
$f'(x)$	-3	-2	1	2	3	4

a) Approximate $\int_0^6 f(x)dx$ by a Riemann sum using the midpoint rule with three rectangles. **(5 pts.)**

a) The intervals are all equal, $\Delta x = 2,$ and the midpoints are x = 1, x = 3, x = 5, so

$$\int_0^6 f(x)dx \cong f(1)\Delta x + f(3)\Delta x + f(5)\Delta x = 2(3+0+4) = 14.$$

b) Approximate $\int_0^6 f(x)dx$ by a Riemann sum by using right endpoints and six rectangles. **(5 pts.)**

b) In this case the intervals are all equal, $\Delta x = 1,$ and the right endpoints are x = 1,…, x = 6, so

$$\int_0^6 f(x)dx \cong f(1)\Delta x + \cdots + f(6)\Delta x = 3+2+0+1+4+5 = 15.$$

c) Find the exact value of $\int_1^4 f'(x)dx$. Show your work and state any theorems that you use. **(5 pts.)**

c) It follows from the fundamental theorem of calculus that

$$\int_1^4 f'(x)dx = f(4) - f(1) = 1 - 3 = -2.$$

8. Solve each of the following:

(a) Evaluate $\int_0^{\pi^2} \frac{\cos(\sqrt{x})dx}{\sqrt{x}}$. **(3 pts.)**

(b) Evaluate $\int \left(xe^{x^2} + \frac{1}{x \ln x} \right) dx$ **(3 pts.)**

(c) Evaluate $\int_0^{\pi} \tan x \cos^2 x dx$. **(3 pts.)**

(d) Evaluate $2\int_0^1 (2-2x^2)dx$. **(2 pts)**

(e) Show that (d) is the area between the curves $y = -x^2$ and $y + 3x^2 = 2$. **(4 pts.)**

Solution 8.

(a) Let $u = \sqrt{x}$, so $\int \frac{\cos(\sqrt{x})dx}{\sqrt{x}} = 2\int \cos u du = -2\sin u + c \Rightarrow \int_0^{\pi^2} \frac{\cos(\sqrt{x})dx}{\sqrt{x}} = -2(\sin \pi - \sin 0) = 0.$

(b) We compute using the substitutions $u = x^2$ and $v = \ln x$ that the integral is equal to

$$\int xe^{x^2} dx + \int \frac{dx}{x \ln x} = \frac{1}{2}\int \left(e^{x^2}\right)' dx + \int \frac{d(\ln x)}{\ln x} = \frac{1}{2}e^{x^2} + \ln(\ln x) + c, \ (x > 0).$$

(c) For this integral,

$$\int_0^{\pi} \tan x \cos^2 x dx = \int_0^{\pi} \sin x \cos x dx = (1/2)\int_0^{\pi} \sin 2x dx = (1/4)[\cos 2x]_0^{\pi} = (1/4)[1-1] = 0.$$

(d) $2\int_0^1 (2-2x^2)dx = 2\left[2x - (2/3)x^3\right]_0^1 = 2[2-(2/3)] = 8/3.$

(e) The area in question is symmetric. To find the intersection points, we set $y = -x^2 = 2 - 3x^2$, which implies that $x = \pm 1$, so by symmetry, we find that the area is

$$2\int_0^1 \left(2 - 3x^2 - (-x^2)\right)dx = 2\int_0^1 \left(2 - 2x^2\right)dx.$$