# Instabilities of two-phase Hele-Shaw flow of complex fluids


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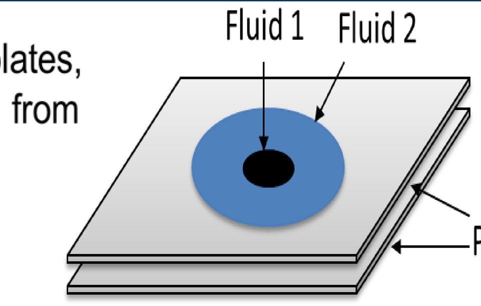
## Abstract


We present the results of semester long project focusing on the instabilities that develop in two-phase flow in Hele-Shaw geometry. Experimentally, we have considered few different fluid combinations: water-glycerol, water-PEO, and water-8CB (nematic liquid crystal flow). The last two combinations are known to exhibit non-Newtonian behavior that influences the pattern formation process. Theoretically, we have carried out linear stability analysis and compared the predictions with the experimental results. Computationally, we have carried out Monte Carlo type of simulations based on Diffusion Limited Aggregation (DLA) approach. We have computed various measures of the emerging patterns, including fractal dimension for both experimental and computational results, and we discuss to which degree non-Newtonian behavior of the considered fluids influences these measures.


## Hele-Shaw Cell and Saffman-Taylor Instability




A Hele-Shaw cell contains two plates, held at constant, small, distance from one another. A fluid, Fluid 2, is sandwiched between the plates. Another fluid, Fluid 1, is injected into the Fluid 2. The boundary between the two fluids in unstable; this is called Saffman-Taylor instability. When two Newtonian fluids are used, the system can be modeled using Laplace's Equation, $\nabla^2 P = 0$, when *P* is pressure.

## Linear Stability

The boundary *($R_1$)* between the less viscous fluid and the larger viscosity fluid would grow as a circle under ideal conditions. However small perturbations from imperfections on the plates, small differences in the gap distance, or other conditions that cause a lack of uniformity cause differences in the growth of smaller viscosity fluid.

The effect of the perturbations *($\eta$)* can be modeled as a Fourier transform where N is the amplitude of the perturbation. The curvature of the interface *($\kappa$)* to the first order is the curvature of the inner circle $\kappa = 1/R_1$, and to the second order is the curvature of the perturbation $\kappa_1 = -\eta + \eta_{\theta\theta}/R_1$.

Assuming that the pressure *(P)* follows Laplace's equation $\nabla^2 P = 0$, with boundary conditions $P|_{R_1} = \gamma\kappa$, where $\gamma$ is the surface tension. Assuming the outer edge of the stationary fluid *($R_2$)* is large enough *($R_2 \gg R_1$)* to be unaffected by the pressure of the fluid being injected we set the boundary condition to be $P|_{R_2} = 0$.

Given Laplace's equation and the preceding boundary conditions, the solution to the first order is

$$P(r,\theta) = \frac{\gamma}{R_1}\left(\frac{1}{\ln\left(\frac{R_1}{R_2}\right)} + (1 - \mathrm{m}^2)\mathrm{r}^{-\mathrm{m}}\eta\right).$$

The velocity of the fluid *(u)*, and therefore, the boundary in a Hele-Shaw cell is expressed by Darcy's Law $u = (b^2/12\mu)\nabla P$, where *b* is the distance between the parallel plates and is the viscosity of the more viscous liquid. Using this equation for the velocity we can calculate the time evolution of the interface, and solve for the change in the perturbation over time

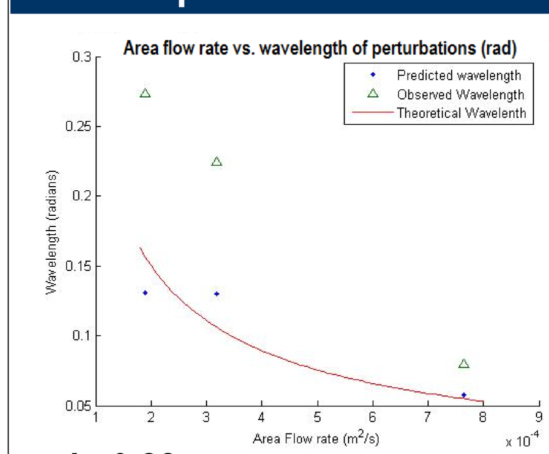
## Linear Stability (cont.)

$$\dot{\eta} = \dot{R}_1\left[-2 + m\left(1 + \frac{\gamma}{R_1}(1 - m^2)\right)\right]\eta.$$

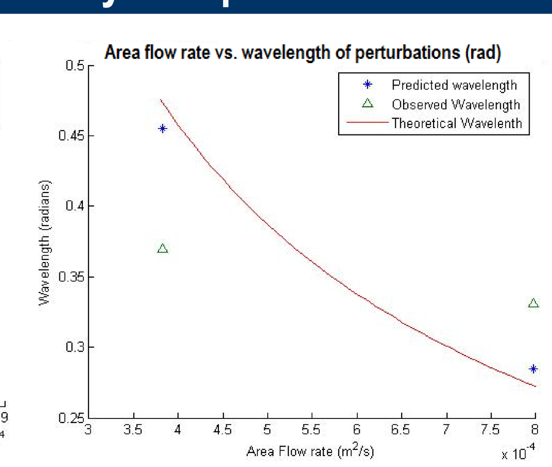
Calling the equation in the square brackets the growth rate *($\sigma$)*, the equation for the perturbation can then be expressed as $\eta(m,t) = \eta(m,0)e^{\sigma t}$. So if $\sigma > 0$ the perturbation is unstable. The effect of an unstable perturbation is growth of perturbations at a faster rate. This effect is called "fingering".

It is noteworthy to include that by assuming surface tension is zero, the Hele-Shaw problem can be simplified and have exact solutions obtainable through complex variables methods (such as conformal mapping). However surface tension is a key to regularizing the problem -- without it, arbitrary solutions can be found and physical applications, such as the distance between fingers, cannot be predicted.

## Comparison of Linear Stability & Experimental Results



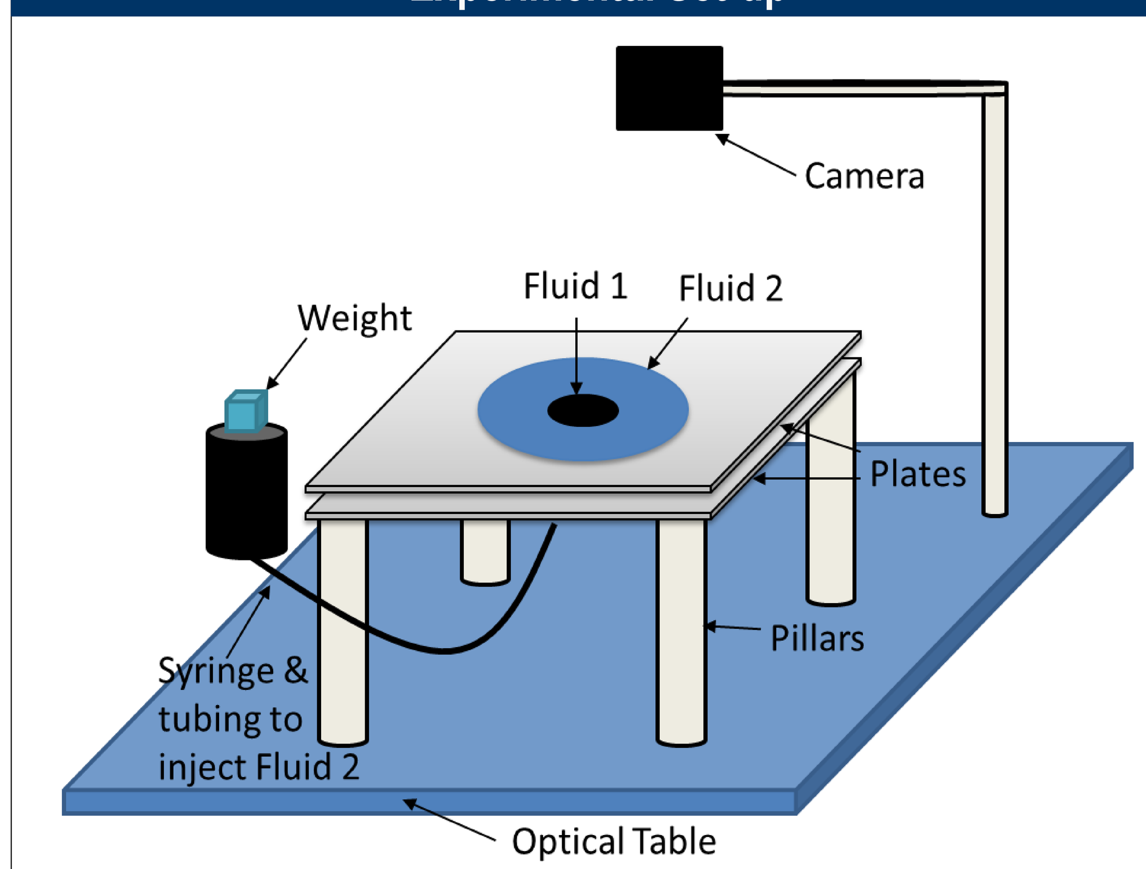

b=0.22 mm




b=0.62 mm

Above we see comparison of linear stability and experimental results as a function of flow rate for two different separation distances between the plates, *b*.

## Experimental Set-up

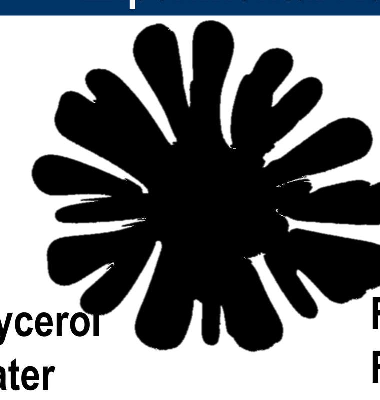



- An optical table was used. Four stand bars where secured in the table, so they could hold up the cell.
- A white piece of paper with a needle sized hole in the center was placed on top of the bars.
- The two plates, identical expect for a needle sized hole in the center of one, where used.
- A piece of tape was placed over the hole in the plate, and that plate was placed on top of the white paper.
- Spacers where placed around the edges of that plate.

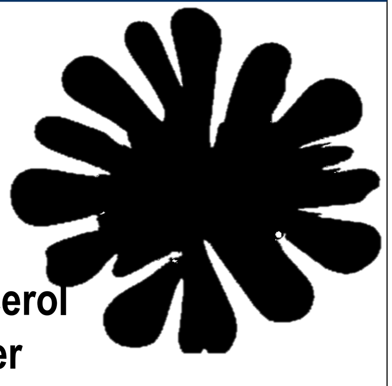
## Experimental Set-up (cont.)

- Fluid 1 was poured in the center of the plate. The other plate was placed on top.
- A syringe, connected to tubing and a needle, was filled with Fluid 2, and the needle was placed in the hole in the bottom plate.
- A camera was mounted above the system to record the results.
- A weight was placed on the plunger of the syringe, and Fluid 2 was injected into Fluid 1.

## Experimental Results: Newtonian

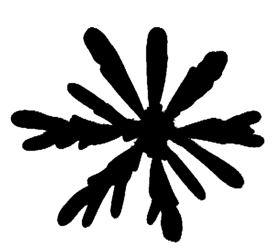


**Fluid 1: Glycerol**
**Fluid 2: Water**
**Spacing: 0.82 mm**
**Fractal Dimension: 1.880**



**Fluid 1: Glycerol**
**Fluid 2: Water**
**Spacing: 0.82 mm**
**Fractal Dimension: 1.865**

## Experimental Results: Non-Newtonian



**Fluid 1: PEO (Polymer Solution)**
**Fluid 2: Water**
**Spacing: 0.82 mm**
**Fractal Dimension1.755**

**Fluid 1: PEO (Polymer Solution)**
**Fluid 2: Water**
**Spacing: 0.22 mm**
**Fractal Dimension: 1.790**

## Monte Carlo Simulations

Monte Carlo Methods can be used to solve Laplace's equation. Diffusion limited aggregation simulations are based on Monte Carlo simulations.

To apply the Monte-Carlo Method to this system, the circular area occupied by a Newtonian fluid is broken down into a grid. A "seed" is placed in the center of the circle; this seed represents that injected Newtonian fluid. Each grid space unoccupied by the seed is given a value of zero. Each grid space occupied by the seed is given a value of one. A number of "walkers" were placed randomly on the circle surrounding the seed. The circle must have a large radius to mimic the walkers coming in to the seed from infinity. Each walker moves randomly, either up, down, left or right. Each direction is equally probable. When the walker hits the seed, there is a probability it will "stick" to the seed. If it sticks, the space is given a value of one and it becomes part of the seed. The probability of the walker sticking depends on the local curvature of the seed,

$$P(N_t) = A\left(\frac{N_t}{l^2} - \frac{l-1}{2l}\right) + B,$$

When A is equivalent to surface tension, and B is an adjustable parameter.[4] The growing seed is model of Fluid 1 as it is injected into Fluid 2.

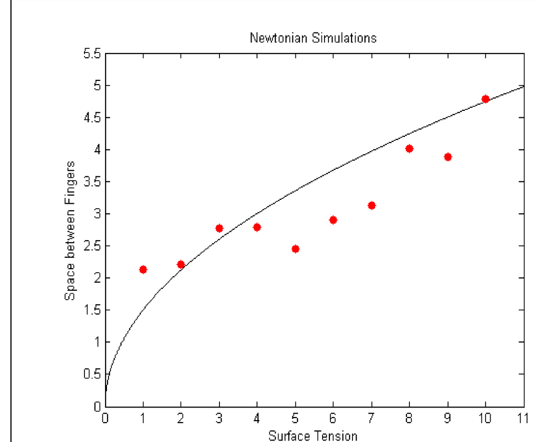
This method leaves the possibility for holes to form in the simulation. A solution to fill these holes is to move the walkers into the holes as they are created. This is accomplished by allowing a walker than has already "stuck" to move into the neighboring grid space that has a lower energy.

## Monte Carlo Simulations (cont.)

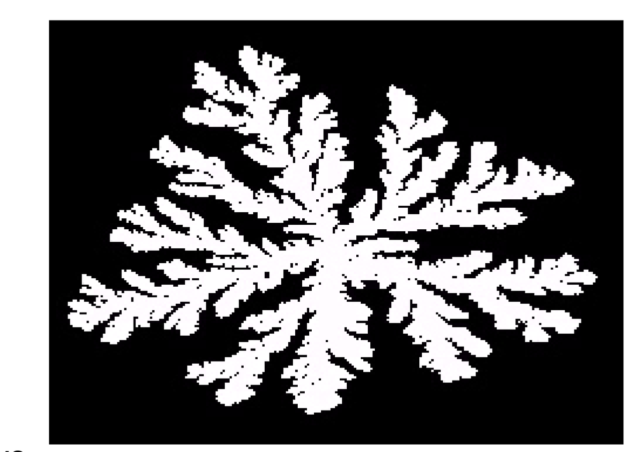
A non-Newtonian fluid in a Hele-Shaw cell reacts differently than a Newtonian fluid. Viscosity determines how difficult it is for the layers of fluid to move across each-other; in other words viscosity determines how the fluid flows. For Newtonian fluids the viscosity is a constant. For non-Newtonian fluids, specifically the Carreau model for shear thinning fluids, viscosity depends on the shear rate and for the Hele-Shaw problem the shear rate depends on the fluid velocity. The idea of velocity is difficult for a DLA simulation; suppose a particle sticks and occupies a cell, it then checks for unoccupied cells above, below, left or right of it. If any of these are occupied and its opposite cell is empty the empty cell is given a velocity number of 1, meaning that the interface is growing fast in that direction. After each new particle sticks increase all of the velocity numbers in the matrix by 1, simulating the interface is growing fast in the cell with the lowest velocity number. Finally, we apply the probability modification that increases the probability the particle will stick based on the parameter $k^{.5}$ which is the current time-step or current number of particles, and C is the velocity number described previously:

$$P = P_0\left(1 + \frac{k^{.5}}{C}\right).$$

## Simulation Results: Newtonian

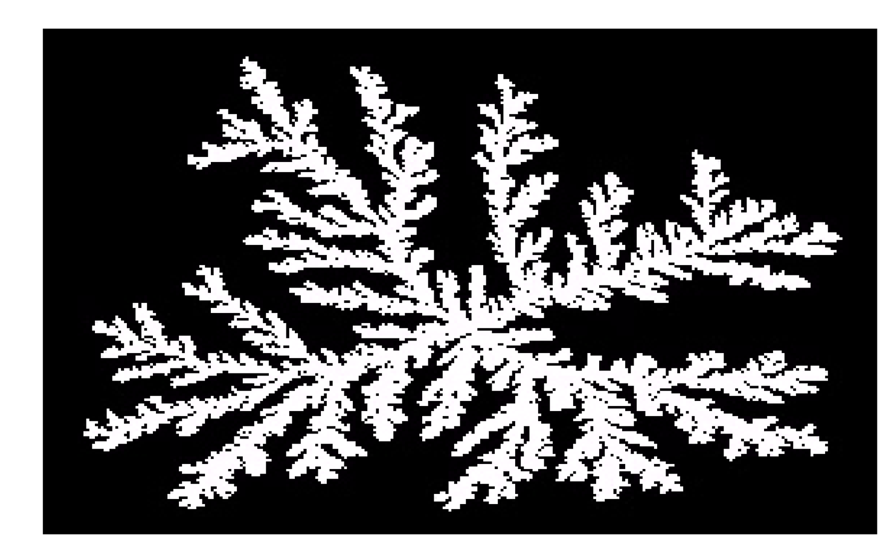



Surface Tension of simulations vs. space between fingers compared to a square root relation as seen in linear stability analysis.



Fractal Dimension: 1.7933

## Simulation Results: Non-Newtonian



Fractal Dimension: 1.6895

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