Show all work and justify all steps of each argument you make.

1) (20 points) Let $A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$ with $v_1 = (1, 2, 3)^T$, $v_2 = (2, 4, 6)^T$, $v_3 = (3, 8, 7)^T$, $v_4 = (5, 12, 13)^T$, and $b = (b_1, b_2, b_3)^T$. Is the system $Ax = b$ consistent for all $b$ in $\mathbb{R}^3$?

b) Find the general solution in the form $x = x_h + p$ of $Ax = (0, 6, -6)^T$.

c) What is the definition of a basis of a vector space? Find bases and dimensions of $\text{Nul}(A)$, $\text{Col}(A)$ and $\text{Row}(A)$. What is the rank of $A$?

d) (5 points extra credit) Find a basis for $(\text{Row}(A))^\perp$.

2) (15 points) Let $u = (1, 1, 0)^T$, $v = (0, 1, 1)^T$ and $V = \text{span}\{v\}$.

a) Find the projection $w = \text{proj}_V u$ of $u$ onto $V$ and its length $\|w\|$.

b) Find $\|u - w\|$ and the distance from $u$ to $V$.

c) What is the angle between $u$ and $v$?

3) (25 points) Let $A = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ with $x_1 = (4, 1, -2)^T$, $x_2 = (0, 3, 2)^T$ and $x_3 = (0, 0, 4)^T$.

a) Find the characteristic equation and the eigenvalues of $A$.

b) Find bases for the corresponding eigenspaces.

c) Diagonalize $A$ (i.e., write it as $A = PDP^{-1}$). Do not compute $P^{-1}$.

d) Show that $\det A = \det D$ and find $\det A^2 A^T$. Is $A$ invertible? Explain.

4) (20) Let $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ with $v_1 = (1, -1, 0)^T$, $v_2 = (2, 0, -2)^T$ and $v_3 = (3, -3, 3)^T$.

a) Show that $\{v_1, v_2, v_3\}$ is a basis for $\text{Col}(A)$.

b) Find an orthogonal basis for $V = \text{Col}(A)$ (Gram-Schmidt).

c) Find the QR factorization of $A$. Express $R$ in terms of $Q$ and $A$, but don’t compute it.

5) (20 points) a) Find the matrix $A$ corresponding to the quadratic form $Q(x) = 3x_1^2 + 10x_1x_2 + 3x_2^2$.

b) Is $Q$ positive definite, negative definite, or indefinite?

c) Orthogonally diagonalize $A$. Find $A^k$ where $k$ is a positive integer.