DOCTORAL QUALIFYING EXAM
Department of Mathematical Sciences
New Jersey Institute of Technology

Statistics Part B: Real Analysis and Statistical Inference
AUGUST 2014

The first three questions are about Real Analysis and the next three questions are about Statistical Inference.

1. (a) State the definition of a measure.
   
   (b) Prove that if \( A \subset B \), then \( \mu(A) \leq \mu(B) \).
   
   (c) Prove that if \( A_1 \subset A_2 \subset A_3 \subset \ldots \), then \( \lim_{j \to \infty} \mu(A_j) = \mu \left( \bigcup_{i=1}^{\infty} A_i \right) \).
   
   (d) Prove that if \( A_1 \supset A_2 \supset A_3 \supset \ldots \) and \( \mu(A_1) < \infty \), then \( \lim_{j \to \infty} \mu(A_j) = \mu \left( \bigcap_{i=1}^{\infty} A_i \right) \).

2. The Hardy-Littlewood-Sobolev inequality reads:
   
   \[
   \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y) \, dx \, dy \right| \leq C \|f\|_p \|g\|_q,
   \]
   where \( p, q > 1, f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n), 0 < \lambda < n \) with \( \frac{1}{p} + \frac{\lambda}{n} + \frac{1}{q} = 2 \), and \( C > 0 \) depending only on \( p, q \) and \( n \).
   
   (a) Show that this inequality cannot hold for any \( 0 < \lambda < n \) such that
   
   \[
   \frac{1}{p} + \frac{\lambda}{n} + \frac{1}{q} \neq 2.
   \]
   
   (b) If \( \lambda = n - 2 \) and \( p = q \), for which values of \( n \) and \( p \) does the Hardy-Littlewood-Sobolev inequality hold?
   
   (c) Is it possible to choose \( p = q = 2 \) in the Hardy-Littlewood-Sobolev inequality?

3. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined as
   
   \[
   f(x) := \begin{cases} \sqrt{1-|x|^2}, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}
   \]
   
   (a) Sketch the graph of this function. Does this function belong to any of these classes (justify your answer): \( C(\mathbb{R}^2), C^1(\mathbb{R}^2), C^\infty(\mathbb{R}^2), C_c(\mathbb{R}^2), C_c^1(\mathbb{R}^2), C_c^\infty(\mathbb{R}^2) \)?
   
   (b) (extra credit) Prove that this function belongs to \( W^{1,p}(\mathbb{R}^2) \) for any \( p \in [1, 2) \).
4. Let $X_1, \ldots, X_n$ be independent identically distributed $N(0, \theta)$ where $0 < \theta < \infty$.

(a) Find a sufficient statistic for $\theta$.

(b) Find an unbiased estimator of $\theta$ based on the above sufficient statistic and find the variance of the estimator.

5. Let $X_1, \ldots, X_n$ be an independent identically distributed sample from the Cauchy distribution with probability density function given by

$$f(x; \theta) = \frac{1}{\pi[1 + (x - \theta)^2]}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(a) Find the Cramer-Rao lower bound for an unbiased estimator of $\theta$.

(b) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ if $\hat{\theta}$ is the maximum likelihood estimator of $\theta$?

6. Let $X_1, \ldots, X_n$ be a random sample from a gamma distribution with parameters $\alpha = 3$ and $\beta = \theta$. Let $H_0 : \theta = 2$ versus $H_1 : \theta > 2$.

(a) Show that there exists a uniformly most powerful test for $H_0$ against $H_1$, determine the statistic $Y$ upon which the test may be based, and indicate the nature of the best critical region.

(b) Find the probability density function of the statistic $Y$ in this problem part (a). If we want a significance level of 0.05, write an equation which can be used to determine the critical region. Let $\gamma(\theta), \theta \geq 2$, be the power function of the test. Express the power function as an integral.