The first three questions are about Linear Algebra and the next three questions are about Numerical Methods.

1. Consider the linear system of equations

\[
\begin{align*}
    x_1 + 2x_2 &= 0 \\
    x_1 + 3x_2 + x_3 &= 1 \\
    x_2 + 2x_3 &= 3
\end{align*}
\]  

(a) Does this system have a unique solution? If so, find it.
(b) Add a fourth equation

\[a_1x_1 + a_2x_2 + a_3x_3 = b\]

Will the new system always have a unique solution? If so, prove it. If not, find the conditions on the \(a_i\) (\(i = 1, 2, 3\)) and \(b\) so that the new system does have a unique solution.

2. Suppose that \(v\) is a nonzero column vector in \(\mathbb{C}^n\) (\(n > 1\)) and the matrix \(A = vv^H/(v^Hv)\).

(a) What are the eigenvalues of \(A\)? Explain.
(b) Is the matrix \(I + A\) (\(I\) is the \(n \times n\) identity matrix) diagonalizable? Explain.
(c) Find the determinant of \(I + A\).
(d) What is \(A^{2014}\)? Explain.

3. (a) Suppose that \(u_1, \ldots, u_n\) and \(v_1, \ldots, v_n\) are orthonormal bases for \(\mathbb{R}^n\). Construct the matrix \(A\) that transforms each \(v_j\) into \(u_j\) to give \(Av_1 = u_1, \ldots, Av_n = u_n\).
(b) Find the SVD of the matrix

\[
A = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

4. (a) Apply the backward Euler method to the problem \(Y' = \lambda Y\) for \(x > 0\) with \(Y(0) = 1\) and \(\lambda\) an arbitrary real constant. Let \(y_h(x_n)\) be the numerical approximation to the true solution evaluated at \(x_n\) with a step size \(h\). Show that the error is

\[
Y(x_n) - yh(x_n) = -\frac{\lambda^2 x_nh^2}{2} + O(h^2).
\]
(b) Apply the trapezoidal method to \(Y' = \lambda Y\) for \(x > 0\) with \(Y(0) = 1\) and \(\lambda\) an arbitrary real constant. Show first that

\[
yh(x_n) = \left(1 + \frac{\lambda h}{2}\right)^n \left(1 - \frac{\lambda h}{2}\right) 
\]

and then show that

\[
Y(x_n) - yh(x_n) = -\frac{\lambda^3 x_n e^{\lambda x_n}}{12} h^2 + O(h^4).
\]
5. Show that the following iteration for root finding is a second-order method:

\[ x_{n+1} = x_n - \frac{f(x_n)}{D(x_n)}, \quad D(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}, \quad n \geq 0. \]

6. Consider the following two-step method for solving the initial value problem \( y' = f(x, y), y(0) = y_0 \):

\[ y_{n+1} = \frac{1}{2} (y_n + y_{n-1}) + \frac{h}{4} \left[ 4y'_{n+1} - y'_n + 3y'_{n-1} \right], \quad n \geq 1 \]

with \( y'_n \equiv f(x_n, y_n) \) and \( h \) is the step size. Show that it is second-order, and find the leading term in the truncation error. Discuss the stability of this two-step method.