Problem 1.

(a) Find all $2 \times 2$ matrices $A$ such that

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

(b) Find all $2 \times 2$ matrices $A$ such that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

Problem 2. Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$ 

(a) Find the four fundamental subspaces (column space, null space, row space and left null space) of $A$.

(b) Find the set of all $3 \times 3$ real matrices that have the same fundamental subspaces as $A$.

Problem 3.

(a) Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$ 

What are the eigenvalues of $A$?

(b) Let $B$ be an $n \times n$ real symmetric matrix with all zeros on the diagonal ($B_{ll}$ for $l = 1, \ldots, n$). Furthermore, suppose that $I + B$ is positive definite. Prove that the largest eigenvalue of $B$ is less than $n - 1$. 

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Problem 4. Let \( X \) and \( Y \) be random variables such that \( E(X^k) \) and \( E(Y^k) \) exist for \( k = 1, 2, 3, \ldots \). If the ratio \( X/Y \) and its denominator \( Y \) are independent, prove that \( E[(X/Y)^k] = E(X^k)/E(Y^k), k = 1, 2, 3, \ldots \).

Problem 5. Let \( X_1, \ldots, X_n \) be a random sample from
\[
f(x; \theta) = \frac{\exp\{-x - \theta\}}{(1 + \exp\{-x - \theta\})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.
\]
Show that the likelihood equation has a unique solution \( \hat{\theta} \) and the solution is a maximum. Assume that the regularity conditions hold. What can one say about the asymptotic properties of the estimator \( \hat{\theta} \)? Derive an \((1 - \alpha)\) large sample confidence interval for \( \theta_0 \) the true parameter.

Problem 6. Let \( X_1, \ldots, X_n \) denote a random sample from a gamma distribution with \( \alpha = 3 \) and \( \beta = \theta \). Let \( H_0 : \theta = 2 \) and \( H_1 : \theta > 2 \). The gamma density with parameters \((\alpha, \beta)\), \(0 < \alpha, \beta < \infty\), is given by
\[
f(x) = \begin{cases} 
\frac{x^{\alpha-1}\exp\{-x/\beta\}}{\Gamma(\alpha)\beta^\alpha}, & 0 < x < \infty, \\
0, & \text{otherwise.}
\end{cases}
\]

(a) Show that there exists a uniformly most powerful test for \( H_0 \) against \( H_1 \), determine the statistics \( Y \) upon which the test may be based, and indicate the nature of the best critical region.

(b) Find the distribution of \( Y \) in Part(a) of this problem. If we want a significance level of 0.05, write an equation which can be used to determine the critical region. Express the power function as an integral.