Notation. In the following problems, \( \mathbb{Q} \) denotes the set of all rational numbers, and \( \mathbb{R} \) denotes the real numbers. Also \( L^1(S) \) and \( L^2(S) \) denotes the measurable functions that are integrable and square integrable respectively over a domain \( S \).

Problem 1. Let \( \{x_1, x_2, \ldots\} = \mathbb{Q} \cap (0, 1) \). Let \( H(x) \) be defined by

\[
H(x) = \begin{cases} 
1 & x > 0 \\
0 & x \leq 0 
\end{cases}.
\]

Let

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} H(x - x_n).
\]

(a) Show that the infinite sum defining \( f(x) \) converges for \( x \in [0, 1] \).
(b) Show that if \( \xi \in \mathbb{Q} \cap (0, 1) \) then \( f(x) \) is discontinuous at \( \xi \).
(c) Show that \( f(x) \) is continuous at \( x = 0 \).

Problem 2. Determine which of the following functions are in \( L^1(\mathbb{R}) \) and which are in \( L^2(\mathbb{R}) \). Carefully explain all your determinations. (Assume the value of the function to be zero at any point where the function is not defined by the given formula.)

(a) \( f(x) = \frac{1}{1 + |x|} \)

(b) \( f(x) = \frac{e^{-|x|}}{|x|^{1/2}} \)

(c) \( f(x) = \sin(1/x) \)

Problem 3. For an integrable function \( f : [0, 2\pi] \to \mathbb{C} \) the “sum” of its Fourier series is defined to be

\[
\lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}_n e^{inx} \quad \text{where} \quad \hat{f}_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{-inx} dx.
\]

(a) Show

\[
\lim_{n \to \infty} \hat{f}_n = 0.
\]

(b) For each function defined below, find the sum of its Fourier series and explain why the result is valid. We denote the characteristic (or indicator) function of a set \( S \) by \( \chi_S(x) \).

(i) \( f(x) = \chi_A(x) \) with \( A = (0, 2\pi) - \mathbb{Q} \)
(ii) \( f(x) = x \)

**Problem 4.** (i) Suppose \( \{X_j : j = 1, 2, \cdots\} \) are pairwise uncorrelated. Show that, for suitable choices of centering and norming constants, either of the conditions

a) \( \sum_{j=1}^{\infty} \text{var}(X_j) = \infty \),

b) \( n^{-2} \sum_{j=1}^{n} \text{var}(X_j) \to 0 \), as \( n \to \infty \),

are sufficient to guarantee that the sequence \( \{X_j : j = 1, 2, \cdots\} \) obeys WLLN (the Weak Law of Large Numbers).

(ii) Consider the probability space \((\Omega, \mathcal{F}, P)\), where \( \Omega = [0,1] \), \( \mathcal{F} = \mathcal{B}[0,1] \), the Borel \( \sigma \)-field over the unit interval, and \( P \) is the Lebesgue measure on \([0,1] \). Consider the random variables,

\[
X_n : \omega \mapsto \omega + \omega_n; \quad n = 1, 2, \cdots
\]

\[
X : \omega \mapsto \omega.
\]

Investigate, in what senses (among \( \overset{P}{\to}, \overset{a.s.}{\to}, \overset{d}{\to} \) and \( \overset{r}{\to} \)), the r.v.s \( X_n \) does or, does not converge to \( X \)?

**Problem 5.** Consider a Markov Chain \( \{X_n : j = 0, 1, 2, \cdots\} \) assuming values in the state space \( S = \{0, 1, 2, \cdots, N\} \) with transition probabilities \( p_{ij} \equiv P(X_{n+1} = j | X_n = i) \) given by, \( p_{00} = p_{NN} = 1 \), and

\[
p_{ij} = \binom{N}{j} \left( \frac{i}{N} \right)^j \left( 1 - \frac{i}{N} \right)^{N-j},
\]

for all \( i, j \in S \setminus \{0, N\} \). Show that \( \{X_n : n = 0, 1, 2, \cdots\} \) and

\[
\left\{ V_n := \frac{X_n(N - X_n)}{(1 - N^{-1})^n}, n = 0, 1, 2, \cdots \right\}
\]

are both martingales. Do these martingales converge a.s. ? Why ?

**6.** Consider the r.v.s \( X_{(n)} := \max(X_1, X_2, \cdots, X_n) \), where \( X_j := F(Y_j); \ j = 1, 2, \cdots \) and \( \{Y_1, Y_2, \cdots\} \) is an i.i.d. sequence with a continuous c.d.f. \( F \).

a) Show that \( X_{(n)} \) converges in probability to 1.

b) Does the convergence also hold in the \( r \)-th mean \( (X_{(n)} \overset{r}{\to} 1) \) for some or, all \( r > 0 \) ?

c) Find the limiting distribution of \( n(1 - X_{(n)}) \) as \( n \to \infty \).